New Permutation Representations of The Braid Group

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Abstract

We give a new infinite family of group homomorphisms from the braid group B_k to the symmetric group S_{mk} for all k and $m \geq 2$. Most known permutation representations of braids are included in this family. We prove that the homomorphisms in this family are non-cyclic and transitive. For any divisor l of m, $1 \leq l < m$, we prove in particular that if $\frac{m}{l}$ is odd then there are $1+\frac{m}{l}$ non-conjugate homomorphisms included in our family. We define a certain natural restriction on homomorphisms $B_k \to S_n$, common to all homomorphisms in our family, which we term *good*, and of which there are two types. We prove that all good homomorphisms $B_k \to S_{mk}$ of type 1 are included in the infinite family of homomorphisms we gave. For $m = 3$, we prove that all good homomorphisms $B_k \to S_{3k}$ of type 2 are also included in this family. Finally, we refute a conjecture made in [MaSu05] regarding permutation representations of braids and give an updated conjecture.

1 Introduction

The study of permutation representations of braids, i.e., homomorphisms from the braid group B_k on k strings to the symmetric group S_n, is closely related to a variety of fields such as algebraic equations with function coefficients, algebraic functions, configuration spaces and their holomorphic maps and more. For more information on these relations see [Arn70a, Arn70b, Arn70c], [GorLin69, GorLin74], [Kal75, Kal76, Kal93], [Kurth97], [Lin71, Lin72b, Lin79, Lin96a, Lin96b, Lin03, Lin04a].

Most results known to date on permutation representations of braids were obtained by Artin ([Ar47]) and by Lin ([Lin04b]). Some results were also obtained by Matei and Suciu in [MaSu05]

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where permutation representations of B_3 and B_4 are discussed and computed (we shall address a conjecture made in that paper later on). Let us also mention the work of Arouch in [Ab07] who has applied an algorithmic point of view.

Lin ([Lin04b]) gives an explicit description of all non-cyclic transitive homomorphisms $B_k \to S_{2k}$ for $k > 6$ up to conjugation. He proves that there are exactly three such homomorphisms up to conjugation. In our work we give a previously unknown infinite family of group homomorphisms $B_k \to S_{mk}$ for all $k > 8$ and $m \geq 2$. In fact, our description of these homomorphisms (up to conjugation), can be seen to be a generalization of Lin's result. Furthermore, we show that this family can be described essentially by a single formula. Interestingly enough, this single formula includes almost all permutation representations known to date. We prove that all homomorphisms in this family are non-cyclic and transitive. Furthermore, for any divisor l of m, $1 \leq l < m$, we prove in particular that if $\frac{m}{l}$ is odd then there are $1 + \frac{m}{l}$ non-conjugate homomorphisms included in our family (in fact, the condition on m and l which guarantees $1 + \frac{m}{l}$ non-conjugate homomorphisms is more general than that - see proposition 3.25). We define a certain natural restriction on homomorphisms $B_k \to S_n$, common to all homomorphisms in our family, defined as follows

We call a homomorphism $\omega \colon B_k \to S_n$ **good** if one of the following holds

1. $supp(\widehat{\sigma}_i) \cap supp(\widehat{\sigma}_j) = \emptyset$ for $1 \leq i, j \leq k-1$ such that $|i - j| > 1$.

2. $supp(\widehat{\sigma}_1) = \cdots = supp(\widehat{\sigma}_{k-1})$

We prove that all transitive non-cyclic good homomorphisms $B_k \to S_{mk}$ of type 1 are included in the infinite family of homomorphisms we gave. For $m = 3$, we prove that all good homomorphisms $B_k \to S_{3k}$ of type 2 are also included in this family.

We state our conjecture that all non-cyclic transitive homomorphisms $B_k \to S_n$ for $n > 2k$ and $k > 8$ are good. This conjecture is based on our findings as well as those of Lin([Lin04b]) : There is only a finite number of known examples for non-cyclic transitive homomorphisms $B_k \to S_n$ which are not good. These examples are given for small values of k and n (see section 4 in [Lin04b]).

Our paper is organized as follows :

In **section 2** we give the definitions and notations which we shall use throughout the article. We define the important notion of (co)retraction and (co)reduction of a homomorphism $B_k \to S_m$ and prove a relevant key result. In **section 3** we present our new family of permutation representations of braids and prove various results regarding it as follows : In **subsection 3.1** we give a new family of homomorphisms $B_k \to S_{mk}$ and prove these are transitive and non-cyclic. We also exhibit how this family generalizes the known family of homomorphisms $B_k \to S_{2k}$ found by Lin in [Lin04b]. In **subsection 3.2** we define our new family of homomorphisms $B_k \to S_{mk}$ and prove that these homomorphisms are well defined, non-cyclic and transitive. In **subsection 3.3** we define the notion of *good* permutation representations of braids, common to the homomorphisms in our family, of which there are two types and we prove some of its properties. In **subsection 3.4** we prove that there are non-trivial conjugacies between the

homomorphisms defined in our new family of homomorphisms $B_k \to S_{mk}$. In particular, for any divisor l of m, $1 \leq l < m$, we prove that if $\frac{m}{l}$ is odd then there are $1 + \frac{m}{l}$ non-conjugate homomorphisms included in our family. In **section 4** we prove that all type 1 good transitive non-cyclic homomorphisms $B_k \to S_{mk}$ are included in the family of homomorphisms we gave in section 3. In **section 5** we prove that all type 2 good transitive non-cyclic homomorphisms $B_k \to S_{3k}$ are included in the family of homomorphisms we gave in section 3. Finally, in **section 6** we give an updated conjecture (updating the conjecture made in [MaSu05]) regarding permutation representations of braid groups and we explain a way to construct homomorphisms $B_k \to S_n$ for $n \neq mk$. We give an infinite family of non-cyclic transitive homomorphisms $B_k \to S_{\frac{k(k-1)}{2}}$ thus refuting a conjecture made in [MaSu05].

2 Definitions and Basic Properties

Let us fix a few definitions and notations (most of our notations are taken from [Lin04b]).

Definition 2.1. *The* ***canonical presentation*** *of the braid group B_k on k strings consists of $k-1$ generators $\sigma_1, \ldots, \sigma_{k-1}$ and relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \geq 2)\,, \tag{2.1}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (1 \leq i \leq k-2)\,. \tag{2.2}$$

•

Notation 2.2. *We denote a conjugate pair of elements g, h of a group G as $g \sim h$. We denote conjugation as usual by $a^b = b^{-1}ab$ and we denote the commutator as usual $[a,b] = aba^{-1}b^{-1}$. The pair g, h is called* **braid-like** *if $ghg = hgh$; in this case we write $g \infty h$. Clearly, $g \infty h$ implies $g \sim h$.* •

Definition 2.3 (**Conjugate homomorphisms**)**.** *Two group homomorphisms $\phi, \psi \colon G \to H$ are called* **conjugate** *if there exists an element $h \in H$ such that $\psi(g) = h^{-1}\phi(g)h$ for all $g \in G$; in this case we write $\phi \sim \psi$ and $\psi = \phi^h$; "$\sim$" is an equivalence relation in $Hom(G,H)$.* •

Definition 2.4 (**Fixed points, Supports and Invariant sets**)**.** *We regard the symmetric group on a set Γ, $\mathbf{S}(\Gamma)$, as acting on Γ from the left. For any element s of $\mathbf{S}(\Gamma)$ let* $\mathbf{Fix}(\mathbf{s})$ *denote the set $\{\gamma \in \Gamma |\, s(\gamma) = \gamma\}$ of all fixed points of s; the complement* $\mathbf{supp}(\mathbf{s}) = \mathbf{\Gamma} \setminus \mathbf{Fix}(\mathbf{s})$ *is called the* **support** *of s. If $\mathfrak{C} = \{s_1, \ldots, s_n\}$ is a set of permutations then we denote $supp(\mathfrak{C}) = \bigcup_{i=1,\ldots,n} supp(s_i)$. For $\Sigma \subseteq \Gamma$ we identify $\mathbf{S}(\Sigma)$ with the subgroup $\{s \in \mathbf{S}(\Gamma) |\, supp(s) \subseteq \Sigma\} \subseteq \mathbf{S}(\Gamma)$. $s, t \in \mathbf{S}(\Gamma)$ are* **disjoint** *if $supp(s) \cap supp(t) = \emptyset$. For $H \subseteq \mathbf{S}(\Gamma)$ a subset $\Sigma \subseteq \Gamma$ is* H-invariant *if $s(\Sigma) = \Sigma$ for all $s \in H$; we denote by* $\mathbf{Inv}(\mathbf{H})$ *the family of all* non-trivial *(i. e., $\neq \emptyset$ and $\neq \Gamma$) H-invariant subsets of Γ.* •

Remark 2.5. *If Γ and Γ' are two disjoint sets, and $A \in \mathbf{S}(\Gamma), B \in \mathbf{S}(\Gamma')$ are two permutations on them, then it will be our convention in this work that the multiplication AB should mean a permutation in $\mathbf{S}(\Gamma \cup \Gamma')$.* •

Notation 2.6. $\mathbf{S}_n$ *denotes the symmetric group of degree n, that is, the permutation group $\mathbf{S}(\boldsymbol{\Delta}_n)$ of the n point set $\boldsymbol{\Delta}_n = \{1, 2, \ldots, n\}$. The* alternating subgroup $\mathbf{A}_n \subset \mathbf{S}_n$ *consists of all even permutations $S \in \mathbf{S}_n$ and coincides with the commutator subgroup $\mathbf{S}'_n$.* •

Definition 2.7. *The* ***canonical epimorphism*** $\mu = \mu_k \colon B_k \to \mathbf{S}_k$ *is defined by*

$$\mu(\sigma_i) = (i, i+1) \in \mathbf{S}_k$$

•

Definition 2.8. *We say that a group homomorphism $\psi \colon G \to \mathbf{S}_n$ is* ***transitive*** *if its image $\psi(G)$ is transitive, i.e. if $\psi(G)$ as a subgroup of $S_n \cong \mathbf{S}(\Delta_n)$ acts transitively on Δ_n (see Definition 2.4 and Notation 2.6). ψ is called* ***cyclic*** *if its image $\psi(G)$ is a cyclic subgroup of S_n.*

•

Let us record the following

Fact 2.9. *Let a, b be any two elements in a group which satisfy $a \infty b$ (i.e. a and b are a braid-like pair) and $[a, b] = 1$, then we have that $b = a^{ba} = a^a = a$.* •

This fact implies the following

Lemma 2.10. *A homomorphism $\omega \colon B_k \to S_n$ is cyclic iff $\varphi(\sigma_i) = \varphi(\sigma_{i+1})$ for some $i \in \{1, \ldots, k-2\}$ iff $\varphi(\sigma_1) = \cdots \varphi(\sigma_{k-1})$.*

Proof. If ω is cyclic then in particular we have that for each $i = 1, \ldots, k-2$

$$[\omega(\sigma_i), \omega(\sigma_{i+1})] = 1$$

but since ω is a homomorphism we also have

$$\omega(\sigma_i) \infty \omega(\sigma_{i+1})$$

and this implies by Fact 2.9 that $\omega(\sigma_i) = \omega(\sigma_{i+1})$ for all $i = 1, \ldots, k-2$.

Conversely, if $\omega(\sigma_i) = \omega(\sigma_{i+1})$ for some $i \in \{1, \ldots, k-2\}$, then $\omega(\sigma_{i+2}) \infty \omega(\sigma_{i+1})$ and $1 = [\omega(\sigma_{i+2}), \omega(\sigma_i)] = [\omega(\sigma_{i+2}), \omega(\sigma_{i+1})]$ implies by Fact 2.9 that $\omega(\sigma_{i+1}) = \omega(\sigma_{i+2})$. Applying the same argument for each consecutive index gives that $\omega(\sigma_1) = \cdots = \omega(\sigma_{k-1})$ and hence $Im(\omega)$ is a cyclic subgroup of S_n generated by $\omega(\sigma_1)$. □

Definition 2.11 (Cyclic types, Components). *Let $A = C_1 \cdots C_q$ be the cyclic decomposition of $A \in \mathbf{S}_n$ and $r_i \geq 2$ be the length of the cycle C_i $(1 \leq i \leq q)$; the* unordered *q-tuple of the natural numbers $[r_1, \ldots, r_q]$ is called the* **cyclic type** *of A and is denoted by $[A]$ (each r_i occurs in $[A]$ as many times as r_i-cycles occur in the cyclic decomposition of A). Clearly,*

$ord(A) = LCM(r_1, \dots, r_q)$ (the least common multiple of $r_1, ..., r_q$). For $A \in \mathbf{S}_n$ and a natural $r \geq 2$ we denote by $\mathfrak{C}_r(A)$ the set of all r-cycles that occur in the cyclic decomposition of A; we call this set the **r - component** *of A. The* **length** *of an r-component $\mathfrak{C}_r(A)$ is the number $|\mathfrak{C}_r(A)|$. The set $Fix(A)$ is called the* **degenerate component** *of A.* •

Notation 2.12. *For a permutation representation $\omega\colon B_k \to S_n$ we shall sometimes denote $\omega(\sigma_i)$ as $\widehat{\sigma}_i$ when ω is understood from the context.* •

Definition 2.13 (**Disjoint product of homomorphisms**). *Given a disjoint decomposition $\mathbf{\Delta}_n = D_1 \cup \dots \cup D_q$, $|D_j| = n_j$, we have the corresponding embedding $\mathbf{S}(D_1) \times \dots \times \mathbf{S}(D_q) \hookrightarrow \mathbf{S}_n$. For a family of group homomorphisms $\psi_j\colon G \to \mathbf{S}(D_j) \cong \mathbf{S}_{n_j}$, $j = 1, ..., q$, we define the* **disjoint product**

$$\psi = \psi_1 \times \dots \times \psi_q\colon G \to \mathbf{S}(D_1) \times \dots \times \mathbf{S}(D_q) \hookrightarrow \mathbf{S}_n$$

by $\psi(g) = \psi_1(g) \cdots \psi_q(g) \in \mathbf{S}_n$, $g \in G$. •

Definition 2.14 (**Reductions of homomorphisms to $\mathbf{S_n}$**). *Let $\omega\colon G \to \mathbf{S}_n$ be a group homomorphism, $H = Im(\omega)$ and $\Sigma \in Inv(H)$ (for instance, Σ may be an H-orbit). Define the homomorphisms*

$$\phi_\Sigma\colon H \ni s \mapsto s|\Sigma \in \mathbf{S}(\Sigma) \quad \text{and} \quad \omega_\Sigma = \phi_\Sigma \circ \omega\colon G \xrightarrow{\omega} H \xrightarrow{\phi_\Sigma} \mathbf{S}(\Sigma)\,;$$

the composition $\omega_\Sigma = \phi_\Sigma \circ \omega$ is called the **reduction of** *ω* **to $\mathbf{\Sigma}$**; *it is transitive if and only if Σ is an $Im(\omega)$-orbit. Any homomorphism ω is the disjoint product of its reductions to all $Im(\omega)$-orbits (this is just the decomposition of the representation ω into the direct sum of irreducible representations).* •

Definition 2.15 (**Retractions and Coretractions**). *Let $\omega\colon B_k \to S_n$ be a homomorphism. Let $\mathfrak{D}_r = \{D_1, \dots, D_t\}$ be an ordered set of r-cycles in the cycle decomposition of $\widehat{\sigma}_1$ for some $r \geq 2$. We call $\mathfrak{D}_r$ an* **r-*subcomponent*** *of $\widehat{\sigma}_1$. Since $\widehat{\sigma}_3, ..., \widehat{\sigma}_{k-1}$ commute with $\widehat{\sigma}_1$, the conjugation of $\widehat{\sigma}_1$ by any of the elements $\widehat{\sigma}_i$ $(3 \leq i \leq k-1)$ induces a certain permutation of the r-cycles in the cycle decomposition of $\widehat{\sigma}_1$; suppose this permutation permutes the cycles of the r-subcomponent $\mathfrak{D}_r$ among themselves, then this gives rise to a homomorphism*

$$\Omega_{\mathfrak{D}_r}\colon B_{k-2} \to \mathbf{S}(\mathfrak{D}_r) \cong \mathbf{S}_t,$$

where B_{k-2} is the subgroup of B_k generated by $\sigma_3, ..., \sigma_{k-1}$ and each σ_i is sent to the permutation it induces on the t r-cycles in $\mathfrak{D}_r$.

The homomorphism $\Omega_{\mathfrak{D}_r}$ is called the ***retraction*** *of ω to the r-subcomponent $\mathfrak{D}_r$.*

Furthermore, since

$$supp(\mathfrak{D}_r) = \bigcup_{i=1,\dots,t} supp(D_i) \in Inv(\{\widehat{\sigma}_3, \dots, \widehat{\sigma}_{k-1}\})$$

(since, according to our assumption, the conjugation of $\widehat{\sigma}_1$ by each permutation in $\{\widehat{\sigma}_3, \dots, \widehat{\sigma}_{k-1}\}$ induces a permutation of the cycles in $\mathfrak{D}_r$ among themselves), we can define the reduction of ω to $supp(\mathfrak{D}_r)$:

$$\omega_{supp(\mathfrak{D}_r)}\colon B_{k-2} \to \mathbf{S}(supp(\mathfrak{D}_r))$$

*where B_{k-2} is the subgroup of B_k generated by $\sigma_3, ..., \sigma_{k-1}$. The homomorphism $\omega_{supp(\mathfrak{D}_r)}$ thus defined is called the **reduction of ω corresponding to the r-subcomponent $\mathfrak{D}_\mathbf{r}$**.*

Similarly, suppose that $\mathfrak{E}_r = \{E_1, \dots, E_t\}$ is an r-subcomponent of $\widehat{\sigma}_{k-1}$ and suppose that $\widehat{\sigma}_1, ..., \widehat{\sigma}_{k-3}$ permute the cycles in $\mathfrak{E}_r$ among themselves then we can define

$$\Omega_{\mathfrak{E}_r}\colon B^*_{k-2} \to \mathbf{S}(\mathfrak{E}_r) \cong \mathbf{S}_t,$$

where B^*_{k-2} is the subgroup of B_k generated by $\sigma_1, ..., \sigma_{k-3}$ and each σ_i is sent to the permutation it induces on the t r-cycles in $\mathfrak{E}_r$.

*The homomorphism $\Omega_{\mathfrak{E}_r}$ is called the **coretraction** of ω to the r-subcomponent $\mathfrak{E}_r$.*

Furthermore, since

$$supp(\mathfrak{E}_r) = \bigcup_{i=1,\dots,t} supp(E_i) \in Inv(\{\widehat{\sigma}_1, \dots, \widehat{\sigma}_{k-3}\})$$

(since, according to our assumption, the conjugation of $\widehat{\sigma}_{k-1}$ with each permutation in $\{\widehat{\sigma}_1, \dots, \widehat{\sigma}_{k-3}\}$ induces a permutation of the cycles in $\mathfrak{E}_r$ among themselves), we can define the reduction of ω to $supp(\mathfrak{E}_r)$:

$$\omega_{supp(\mathfrak{E}_r)}\colon B^*_{k-2} \to \mathbf{S}(supp(\mathfrak{E}_r))$$

*where B^*_{k-2} is the subgroup of B_k generated by $\sigma_1, ..., \sigma_{k-3}$. The homomorphism $\omega_{supp(\mathfrak{E}_r)}$ thus defined is called the **coreduction of ω corresponding to the r-subcomponent $\mathfrak{E}_\mathbf{r}$***

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We will use the following results (based on sections 5 and 6 in [Lin04b])

Proposition 2.16. *Let $\omega\colon B_k \to S_n$ be a homomorphism. Let $\mathfrak{D}_r$ be an r-subcomponent of $\widehat{\sigma}_1$ with t cycles. Let $\Omega_{\mathfrak{D}_r}$ be the retraction of ω to the r-subcomponent $\mathfrak{D}_r$ (assuming it is well defined) and let $\omega_{supp(\mathfrak{D}_r)}$ be the corresponding reduction as defined in Definition 2.15. Then there exists a commuting diagram as follows*

$$\begin{tikzcd}
 & B_{k-2} \arrow[d, "\omega_{supp(\mathfrak{D}_r)}"'] \arrow[dr, "\Omega_{\mathfrak{D}_r}"] & \\
H \arrow[r, "i"] & G \arrow[r, "\pi"] & \mathbf{S}(\mathfrak{D}_r) \cong S_t
\end{tikzcd} \tag{2.3}$$

where B_{k-2} is the subgroup of B_k generated by $\{\sigma_3, \ldots, \sigma_{k-1}\}$, G is a finite group, H is a finite abelian group and the horizontal line in (2.3) *is an exact sequence.*

Similarly, let $\mathfrak{E}_r$ be an r-subcomponent of $\widehat{\sigma}_{k-1}$ with t cycles. Let $\Omega_{\mathfrak{E}_r}$ be the coretraction of ω to the r-subcomponent $\mathfrak{E}_r$ (assuming it is well defined) and let $\omega_{supp(\mathfrak{E}_r)}$ be the corresponding coreduction as defined in Definition 2.15. Then there exists a commuting diagram as follows

$$\begin{tikzcd} & B^*_{k-2} \arrow[d, "\omega_{supp(\mathfrak{E}_r)}"'] \arrow[dr, "\Omega_{\mathfrak{E}_r}"] & \\ H^* \arrow[r, "i"] & G^* \arrow[r, "\pi"] & \mathbf{S}(\mathfrak{E}_r) \cong S_t \end{tikzcd} \tag{2.4}$$

where B^*_{k-2} is the subgroup of B_k generated by $\{\sigma_1, \ldots, \sigma_{k-3}\}$, G^* is a finite group, H^* is a finite abelian group and the horizontal line in (2.4) *is an exact sequence.*

Proof. Let us first prove the claim for diagram (2.3).

Denote the t r-cycles in $\mathfrak{D}_r$ as $\{D_1, \ldots, D_t\}$ and denote $\Sigma = supp(\mathfrak{D}_r)$. As usual, we denote $\omega(\sigma_i)$ as $\widehat{\sigma}_i$ for each i, $1 \le i \le k-1$.

Let G be the centralizer of the element $\mathcal{D} = D_1 \cdots D_t$ in $\mathbf{S}(\Sigma)$.Since any element $g \in G$ commutes with the element $\mathcal{D} = D_1 \cdots D_t$, we have

$$D_1 \cdots D_t = g^{-1} \cdot D_1 \cdots D_t \cdot g = g^{-1} D_1 g \cdots g^{-1} D_t g \qquad \text{for any} \quad g \in G, \tag{2.5}$$

in (2.5) we have that $D_1, \ldots, D_t$ are disjoint r-cycles as well as $g^{-1}D_1g, \ldots, g^{-1}D_tg$. Hence there is a unique permutation $\tau_g \in \mathbf{S}(\mathfrak{D}_r)$ such that $g^{-1}D_jg = \tau_g(D_j)$ for all j. Let us define $\pi\colon G \to \mathbf{S}(\mathfrak{D}_r) \cong S_t$ as follows

$$\pi(g) = \tau_g \quad \text{where} \quad \tau_g(D_i) = g^{-1}D_ig. \tag{2.6}$$

Let $H \cong (\mathbb{Z}_r)^t$ be the abelian subgroup of $\mathbf{S}(\Sigma)$ generated by the r-cycles $\{D_1, \ldots, D_t\}$. Note that since each D_i commutes with $\mathcal{D}$ we have that $H \subset G$. We will now prove that $H = Ker(\pi)$:

Suppose $\pi(g) = id$ for some $g \in G$. Then it follows from (2.6) that each D_i commutes with g and according to Lemma 5.3 this means that $g|_{supp(D_i)}$ is equal to some power of D_i, i.e., $g \in H$.

Conversely, if $h = \prod_j D_j^{c_j} \in H$ where $c_j \in \mathbb{N}$, $0 \le c_j \le r-1$, then for each $i = 1, \ldots, k-1$ we have

$$h^{-1}D_ih = (\prod_j D_j^{c_j})^{-1} D_i \prod_j D_j^{c_j} = D_i$$

so $h \in Ker(\pi)$. Hence $H = Ker(\pi)$ is an abelian normal subgroup of G.

Thereby, we obtain the exact sequence

$$H \xrightarrow{i} G \xrightarrow{\pi} \mathbf{S}(\mathfrak{D}_r) \cong S_t \tag{2.7}$$

where i is the embedding of H in G and π is as defined in (2.6).

Consider the following diagram, which includes (2.7)

$$\begin{tikzcd} & B_{k-2} \arrow[d, "\omega_{supp(\mathfrak{D}_r)}"'] \arrow[dr, "\Omega_{\mathfrak{D}_r}"] & \\ H \arrow[r, "i"] & G \arrow[r, "\pi"] & \mathbf{S}(\mathfrak{D}_r) \cong S_t \end{tikzcd} \tag{2.8}$$

First note that the image of $\omega_{supp(\mathfrak{D}_r)}$ indeed lies in G : For each i, $3 \le i \le k-1$, we have by Definition 2.15 that $\omega_{supp(\mathfrak{D}_r)}(\sigma_i)$ is equal to $\widehat{\sigma}_i|_\Sigma$ and that conjugation of $\widehat{\sigma}_1$ by $\widehat{\sigma}_i|_\Sigma$ induces a permutation of the r-cycles of $\mathfrak{D}_r$ which means in particular that each $\widehat{\sigma}_i|_\Sigma$ commutes with $\mathcal{D} = D_1 \cdots D_t$, hence $Im(\omega_{supp(\mathfrak{D}_r)}) \subseteq G$.

Furthermore, by Definition 2.15, $\Omega_{\mathfrak{D}_r}$ send σ_i, $3 \le i \le k-1$, to the permutation that is induced on the r-cycles in $\mathfrak{D}_r$ by conjugating $\widehat{\sigma}_1$ with $\widehat{\sigma}_i$ or, what is equivalent, by conjugating $\widehat{\sigma}_1$ with $\widehat{\sigma}_i|_\Sigma$. But this is exactly $\pi \circ \omega_{supp(\mathfrak{D}_r)}(\sigma_i)$ according to the definition of π in (2.6). This means that the diagram (2.8) is commutative.

Now if $\mathfrak{E}_r = \{E_1, \ldots, E_t\}$ is an r-subcomponent of $\widehat{\sigma}_{k-1}$ with t cycles and $\Omega_{\mathfrak{E}_r}$ is the coretraction of ω to the r-subcomponent $\mathfrak{E}_r$ and $\omega_{supp(\mathfrak{E}_r)}$ is the corresponding coreduction as defined in Definition 2.15, then the analogous claim about (2.4) is proved in a completely similar way where in this case $H^* \cong (\mathbb{Z}_r)^t$ is the subgroup of $\mathbf{S}(supp(\mathfrak{E}_r))$ generated by the cycles of $\mathfrak{E}_r$ and G^* is the centralizer of the element $\mathcal{E} = E_1 \cdots E_t$ in $\mathbf{S}(supp(\mathfrak{E}_r))$.

□

Corollary 2.17. *Let $\omega\colon B_k \to S_n$ be a homomorphism. If the (co)retraction of ω to an r-subcomponent for some $r \ge 2$ is cyclic then the (co)reduction of ω to that r-subcomponent is cyclic as well.* •

Proof. Let $\mathfrak{D}_r$ be an r-subcomponent of $\widehat{\sigma}_1$ with t cycles. Denote the t r-cycles in $\mathfrak{D}_r$ as $\{D_1, \ldots, D_t\}$. Let $\Omega_{\mathfrak{D}_r}$ be the retraction of ω to the r-subcomponent $\mathfrak{D}_r$ (assuming it is well defined) and let $\omega_{supp(\mathfrak{D}_r)}$ be the corresponding reduction as defined in Definition 2.15.

For the rest of the proof let us denote $\Sigma = supp(\mathfrak{D}_r)$. According to Proposition 2.16 we have the following commuting diagram

$$\begin{tikzcd}
 & B_{k-2} \arrow[d, "\omega_\Sigma"'] \arrow[dr, "\Omega_{\mathfrak{D}_r}"] & \\
H \arrow[r, "i"] & G \arrow[r, "\pi"] & \mathbf{S}(\mathfrak{D}_r) \cong S_t
\end{tikzcd} \tag{2.9}$$

where B_{k-2} is the subgroup of B_k generated by $\{\sigma_3, \ldots, \sigma_{k-1}\}$, G is a finite group, H is a finite abelian group and the horizontal line in (2.9) is an exact sequence.

Denote the commutator subgroup of B_{k-2} as B'_{k-2}. Suppose $\Omega_{\mathfrak{D}_r}$ is cyclic. In particular, the image of $\Omega_{\mathfrak{D}_r}$ is abelian. Hence, since the homomorphic image of any commutator in an abelian group is trivial, we have that $\Omega_{\mathfrak{D}_r}(B'_{k-2}) = \{1\}$. By Proposition 2.16 $\Omega_{\mathfrak{D}_r} = \pi \circ \omega_\Sigma$ and so $\pi \circ \omega_\Sigma(B'_{k-2}) = \{1\}$. Hence $\omega_\Sigma(B'_{k-2})$ is contained in the abelian group $Ker(\pi) = H$ and again since the homomorphic image of any commutator in an abelian group is trivial, we have that $\omega_\Sigma(B'_{k-2}) = \{1\}$. But since $B_{k-2}/B'_{k-2} \cong \mathbb{Z}$ (see section 1.3 in [Lin04b]), this means that ω_Σ factors through $B_{k-2}/B'_{k-2} \cong \mathbb{Z}$, i.e. ω_Σ is cyclic.

The dual claim regarding the coretractions and their corresponding coreduction follows by the same arguments applied to an r-subcomponent of $\widehat{\sigma}_{k-1}$ and using (2.4) of Proposition 2.16.

$\square$

3 New Permutation Representations of the Braid Group

3.1 Model and Standard Permutation Representations

Permutation representations of braids were first studied by Artin in [Ar47b] where he proves

ARTIN THEOREM *Let $\psi\colon B_k \to \mathbf{S}_k$ be a non-cyclic transitive homomorphism. Denote $\alpha = \sigma_1\sigma_2\cdots\sigma_{k-1}$. ($\alpha$ and σ_1 generate B_k, see [Lin04b])*

a) If $k \neq 4$ and $k \neq 6$ then ψ is conjugate to the canonical epimorphism $\mu = \mu_k$.

b) If $k = 6$ and $\psi \not\sim \mu_6$ then ψ is conjugate to the homomorphism ν_6 defined by

$$\nu_6(\sigma_1) = (1,2)(3,4)(5,6), \qquad \nu_6(\alpha) = (1,2,3)(4,5). \tag{3.1}$$

c) If $k = 4$ and $\psi \not\sim \mu_4$ then ψ is conjugate to one of the three homomorphisms $\nu_{4,1}$, $\nu_{4,2}$ and $\nu_{4,3}$ defined by

$$\begin{array}{lll}
\nu_{4,1}(\sigma_1) = (1,2,3,4), & \nu_{4,1}(\alpha) = (1,2); & [\nu_{4,1}(\sigma_3) = \nu_{4,1}(\sigma_1)] \\
\nu_{4,2}(\sigma_1) = (1,3,2,4), & \nu_{4,2}(\alpha) = (1,2,3,4); & [\nu_{4,2}(\sigma_3) = \nu_{4,2}(\sigma_1^{-1})] \\
\nu_{4,3}(\sigma_1) = (1,2,3), & \nu_{4,3}(\alpha) = (1,2)(3,4); & [\nu_{4,3}(\sigma_3) = \nu_{4,3}(\sigma_1)].
\end{array} \tag{3.2}$$

d) ψ *is surjective except for the case when* $k=4$, $\psi \sim \nu_{4,3}$ *and* $Im(\psi) = \mathbf{A}_4$.

■

The next important work on the subject was done in [Lin04b]. Among other results, Lin classifies homomorphisms $B_k \to S_{2k}$ for $k > 6$ and we quote this result here

Definition 3.1. *The following three homomorphisms* $\varphi_1, \varphi_2, \varphi_3 \colon B_k \to \mathbf{S}_{2k}$ *are called the* ***model*** *ones:*

$$\varphi_1(\sigma_i) = \underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{4\text{-}cycle};$$

$$\varphi_2(\sigma_i) = (1,2)(3,4)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+1)(2i, 2i+2)}_{two\ transpositions} \times$$
$$\times\,(2i+3, 2i+4)\cdots(2k-1, 2k);$$

$$\varphi_3(\sigma_i) = (1,2)(3,4)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{4\text{-}cycle} \times$$
$$\times\,(2i+3, 2i+4)\cdots(2k-1, 2k);$$

in each of the above formulas $i = 1, ..., k-1$. *A homomorphism* $\psi \colon B_k \to \mathbf{S}_{2k}$ *is said to be* ***standard*** *if it is conjugate to one of the three model homomorphisms* φ_1, φ_2, φ_3. •

We will use the following

Lemma 3.2. *Let* $\phi \colon B_k \to S_{2k}$ *be a standard homomorphism which is conjugate to either of the model homomorphisms* φ_2 *or* φ_3 *as defined in Definition 3.1. If there exist permutations* $R, R_1, \ldots, R_{k-1} \in S_{2k}$ *with* R *disjoint from* $R_1, \ldots, R_{k-1}$, *such that*

$$\phi(\sigma_i) = R \cdot R_i \qquad \textit{for} \quad i = 3, \ldots, k-1$$

or

$$\phi(\sigma_i) = R \cdot R_i \qquad \textit{for} \quad i = 1, \ldots, k-3$$

then $|supp(R)| \le 4$.

Proof. Let us prove the claim in case ϕ is conjugate to φ_2 and $i = 3, \ldots, k-1$. The other three cases (ϕ is conjugate to φ_3 or $i = 1, \ldots, k-3$) follow in a completely similar way.

First assume $\phi = \varphi_2$. Recall the definition of this model homomorphism (see Definition 3.1)

$$\varphi_2(\sigma_i) = (1,2)(3,4)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+1)(2i, 2i+2)}_{\text{two transpositions}} \times$$
$$\times\,(2i+3, 2i+4)\cdots(2k-1, 2k);$$

We see that

$$\varphi_2(\sigma_i) = (1,2)(3,4) \cdot S_i \qquad \text{for} \quad i = 3, \ldots, k-1 \tag{3.3}$$

where

$$S_i = (5,6)(7,8) \cdot (9,10) \cdots (2i-3, 2i-2) \cdot (2i-1, 2i+1)(2i, 2i+2) \cdot \\ \cdot (2i+3, 2i+4) \cdots (2k-1, 2k)$$

for $i = 3, \ldots, k-1$.

Now suppose that

$$\varphi_2(\sigma_i) = R \cdot R_i \qquad \text{for} \quad i = 3, \ldots, k-1 \tag{3.4}$$

for some $R, R_i \in S_{2k}$ where R is disjoint from the R_i. We claim that R is disjoint from each S_i in (3.3) for $i = 3, \ldots, k-1$. For suppose, by way of contradiction, that $supp(R) \cap supp(S_j) \neq \emptyset$ for some j,$3 \leq j \leq k-1$. Then equating (3.3) and (3.4) for $i = j$ gives $R \cdot R_j = (1,2)(3,4)S_j$ and so the cyclic decomposition of R contains some cycle in the cyclic decomposition of S_j. This means that the cyclic decomposition of R contains some cycle of the form $(l, l+1)$ or $(l, l+2)$ for some l, $5 \leq l \leq 2k-1$. Now if the cyclic decomposition of R contains $(l, l+1)$ then equating (3.3) and (3.4) for $l' = min(\lfloor \frac{l+1}{2} \rfloor, k-1)$ gives a contradiction since the cyclic decomposition of $S_{l'}$ does not contain $(l, l+1)$. Furthermore, if the cyclic decomposition of R contains $(l, l+2)$ then equating (3.3) and (3.4) for any l'', $3 \leq l'' \leq k-1$, such that $|l'' - l'| > 1$, gives a contradiction since the cyclic decomposition of $S_{l''}$ does not contain $(l, l+2)$.

We conclude that R in (3.4) is disjoint from each S_i in (3.3) and by equating (3.3) and (3.4) we see that each cycle in the cyclic decomposition of R must be included in the cyclic decomposition of $(1,2)(3,4)$. Hence R could be only one of four possibilities : id, $(1,2)$, $(3,4)$ or $(1,2)(3,4)$. In any case $|supp(R)| \leq 4$.

Now suppose ϕ is conjugate to φ_2 and

$$\phi(\sigma_i) = R \cdot R_i \qquad \text{for} \quad i = 3, \ldots, k-1$$

for some $R, R_1, \ldots, R_{k-1} \in S_{2k}$ where R is disjoint from $R_1, \ldots, R_{k-1}$. By definition there exists a $C \in S_{2k}$ such that $\varphi_2 = \phi^C$, so we can write

$$\varphi_2(\sigma_i) = R^C \cdot R_i^C \qquad \text{for} \quad i = 3, \ldots, k-1$$

but clearly R^C is disjoint from the R_i^C and according to the argument above we have that $|supp(R^C)| \leq 4$ and so necessarily $|supp(R)| \leq 4$ which concludes the proof.

□

The following results are proved in [Lin04b], we give the references within.

Theorem 3.3. *a) For $6 < k < n < 2k$ every transitive homomorphism $B_k \to \mathbf{S}_n$ is cyclic (Theorem F (a) in [Lin04b]).*

b) For $k > 6$ every non-cyclic transitive homomorphism $\psi\colon B_k \to \mathbf{S}_{2k}$ is standard (Theorem F (b) in [Lin04b]).

c) For $6 < k < n < 2k$ every non-cyclic homomorphism $\psi\colon B_k \to \mathbf{S}_n$ is conjugate to a homomorphism of the form $\mu_k \times \widetilde{\psi}$, where $\widetilde{\psi}\colon B_k \to \mathbf{S}_{n-k}$ is a cyclic homomorphism (Theorem 7.26 (b) in [Lin04b]).

d) for $k > 6$ every non-cyclic homomorphism $\psi\colon B_k \to \mathbf{S}_k$ is conjugate to μ_k (Theorem 3.9 in [Lin04b]).

e) for $k > 4$ and $n < k$ every homomorphism $\psi\colon B_k \to \mathbf{S}_n$ is cyclic (Theorem A (a) in [Lin04b]).

3.2 New Model and Standard Permutation Representations

In this subsection we define new model and standard permutation representations of the braid group, extending those defined in Definition 3.1. We begin with a few notations and preliminary results.

Notation 3.4. *Throughout the paper we shall use the following notation : For each $m, i \in \mathbb{N}$, we denote by A_i^m the following permutation*

$$A_i^m = (1 + m(i-1), \ldots, mi) \tag{3.5}$$

so each A_i^m is an m-cycle in $\mathbf{S}(\{1 + m(i-1), \ldots, mi\})$. We shall sometimes denote A_i^m as A_i when m is understood from the context. Also, we denote for each j, $0 \le j \le m-1$

$$a_j^i = 1 + m(i-1) + j \in supp(A_i^m)$$

Furthermore, we denote $\triangledown_m = \{0, \ldots, m-1\} \subset \mathbb{Z}$.

Finally, for $a \in \mathbb{Z}$ we shall denote by $|a|_m \in \triangledown_m$ the residue of a modulo m (e.g. $|8|_5 = 3 \in \triangledown_5 \subset \mathbb{Z}$). •

We now prove

Lemma 3.5. *Let $2 \le m, r \in \mathbb{N}$ and let $\sigma \in S_r$ be some permutation, then for every choice of r elements*

$$t_i \in \bigtriangledown_m \quad 1 \le i \le r$$

we can define an element $C \in \mathbf{S}(\cup_{i=1\ldots r} supp(A_i^m)) \cong S_{mr}$ *such that*

$$(A_i^m)^{C^{-1}} = A_{\sigma(i)}^m \quad 1 \le i \le r \tag{3.6}$$

by the formula

$$C(a_j^i) = a_{|j+t_{\sigma(i)}|_m}^{\sigma(i)}, \quad a_j^i = 1 + m(i-1) + j \in Supp(A_i) \tag{3.7}$$

for each $1 \le i \le r$ *and* $j \in \bigtriangledown_m$.

If σ *is an* r*-cycle, the cyclic decomposition of* C *consists of exactly* $\frac{m}{l}$ lr*-cycles where* l *is the minimal number between* 1 *and* m *such that*

$$l \cdot \sum_{j=0}^{r-1} t_{\sigma^j(i)} (= l \cdot \sum_{j=0}^{r-1} t_j) \equiv 0 (mod\ m).$$

Hence in this case $\sum_{j=0}^{r-1} t_{\sigma^j(i)}$ *determines the cycle structure of* C.

Furthermore, each $C \in \mathbf{S}(\cup_{i=1\ldots r} supp(A_i^m))$ *which satisfies equation 3.6 is given by the formula 3.7 for some choice of* $t_i \in \bigtriangledown_m$.

Proof. To show that equation 3.7 defines C as a permutation in S_{mr} it is enough to show that it is injective as a map from Δ_{mr} to Δ_{mr}. Assume then that $C(a_{j_1}^{i_1}) = C(a_{j_2}^{i_2})$, by equation 3.7 this means that

$$a_{|j_1+t_{\sigma(i_1)}|_m}^{\sigma(i_1)} = a_{|j_2+t_{\sigma(i_2)}|_m}^{\sigma(i_2)}$$

so necessarily $i_1 = i_2$ and

$$j_1 + t_{\sigma(i_1)} \equiv j_2 + t_{\sigma(i_2)} (\text{mod } m)$$

which means that

$$j_1 \equiv j_2 (\text{mod } m)$$

but since $j_1, j_2 \in \bigtriangledown_m$ this means that $j_1 = j_2$ and so C is indeed a well defined permutation.

To see the cyclic decomposition of C in case σ is an r-cycle, note that for any $n \in \mathbb{N}$

$$C^n(a_j^i) = a^{\sigma^n(i)}_{|j+t_{\sigma(i)}+\cdots+t_{\sigma^n(i)}|_m}$$

hence, since $\sigma \in S_r$ is an r-cycle, we have that

$$C^r(a_j^i) = a^{i}_{|j+t_i+t_{\sigma(i)}+\cdots+t_{\sigma^{r-1}(i)}|_m}$$

and

$$C^{lr}(a_j^i) = a^{i}_{|j+l(t_i+t_{\sigma(i)}+\cdots+t_{\sigma^{r-1}(i)})|_m}$$

hence the minimal n such that $C^n(a_j^i) = a_j^i$ is $n = rl$ where l is the minimal number between 1 and m such that

$$l(t_i + t_{\sigma(i)} + \cdots + t_{\sigma^{r-1}(i)}) \equiv 0 (\text{mod } m)$$

so the cyclic decomposition of C consists of exactly $\frac{m}{l}$ lr-cycles.

Now let $C \in S_{mr}$ satisfy the following equation

$$(A_i^m)^{C^{-1}} = A^m_{\sigma(i)} \quad 1 \leq i \leq k$$

then clearly we have that for each $1 \leq i \leq r$ there exists some $t_i \in \bigtriangledown_m$ such that

$$C(a_0^i) = a^{\sigma(i)}_{|t_{\sigma(i)}|_m}$$

but then necessarily

$$C(a_j^i) = a^{\sigma(i)}_{|j+t_{\sigma(i)}|_m}$$

$\square$

Notation 3.6. *Let $\sigma \in \mathbf{S}(\{j, \ldots, j+r-1\}) \cong S_r$ be a permutation and let $t_1, \ldots, t_r \in \bigtriangledown_m$. We denote by $C^{\sigma,m}_{t_1,\ldots,t_r}$ the permutation in $\mathbf{S}(\{\cup_{i=j\ldots j+r-1} supp(A_i^m)\}) \cong S_{mr}$ which is given by the following formula (see notation 3.4)*

$$C^{\sigma,m}_{t_1,\ldots,t_r}(a_q^i) = a^{\sigma(i)}_{|q+t_{\sigma(i)}|_m}, \quad a_q^i = 1 + m(i-1) + q \in Supp(A_i^m) \tag{3.8}$$

for each $j \leq i \leq j+r-1$ *and* $q \in \bigtriangledown_m$.

(that (3.8) *defines a permutation - see remark 3.7).*

$\bullet$

Remark 3.7. *To see that* (3.8) *indeed defines a permutation we have to show that* $C^{\sigma,m}_{t_1,\ldots,t_r}(a^i_q)$ *as a function on* $\cup_{i=j\ldots j+r-1} supp(A^m_i)$ *is injective. But if*

$$C^{\sigma,m}_{t_1,\ldots,t_r}(a^{i_1}_{q_1}) = C^{\sigma,m}_{t_1,\ldots,t_r}(a^{i_2}_{q_2})$$

then by (3.8) *this means that*

$$a^{\sigma(i_1)}_{|q_1+t_{\sigma(i_1)}|_m} = a^{\sigma(i_2)}_{|q_2+t_{\sigma(i_2)}|_m}$$

and so $\sigma(i_1) = \sigma(i_2)$ *which implies that* $i_1 = i_2$ *(since* σ *is a permutation). Hence let us denote* $i = i_1 = i_2$. *So we now have that*

$$|q_1 + t_{\sigma(i)}|_m = |q_2 + t_{\sigma(i)}|_m$$

which means that $|q_1|_m = |q_2|_m$, *but since* $q_1, q_2 \in \bigtriangledown_m$, *it follows that* $q_1 = q_2$, *and so* $a^{i_1}_{q_1} = a^{i_2}_{q_2}$.

Let also us remark on the special case $m = 1$ *in Notation 3.6. If* $\sigma \in \mathbf{S}(\{j,\ldots,j+r-1\}) \cong S_r$ *is an* r*-cycle then*

$$(A^1_i)^{C^{\sigma,1}_{t_j,\ldots,t_{j+r-1}}} = A^1_{\sigma(i)}, \quad j \leq i \leq j+r-1. \tag{3.9}$$

But $A^1_i = (i)$ *and* $t_i \in \bigtriangledown_1 = \{0\}$ *so we can rewrite* (3.9) *as*

$$(i)^{C^{\sigma,1}_{0,\ldots,0}} = \sigma(i), \quad j \leq i \leq j+r-1$$

which means that

$$C^{\sigma,1}_{0,\ldots,0} = \sigma.$$

$\bullet$

Example 3.8. *Let us give an example for an application of Lemma 3.5. Choosing* $m = 6, r = 4$ *and* $\sigma = (1,3,2,4) \in S_4$ *we have*

$$A_1^6 = (1,2,3,4,5,6)$$
$$A_2^6 = (7,8,9,10,11,12)$$
$$A_3^6 = (13,14,15,16,17,18)$$
$$A_4^6 = (19,20,21,22,23,24)$$

now choose $t_1 = 2, t_2 = 1, t_3 = 0, t_4 = 5$. *Since* $t_1+t_2+t_3+t_4 \equiv 2(mod\ 6)$, *and since the minimal* l *(between* 1 *and* $m = 6$*) such that* $2l \equiv 0(mod\ 6)$ *is* $l = 3$, *we expect the cycle decomposition of* $C^{\sigma,6}_{2,1,0,5}$ *to consist of exactly* $(\frac{m}{l} = \frac{6}{3} =)2$ $(rl =)12$*-cycles. Indeed*

$$C^{(1,3,2,4),6}_{2,1,0,5} = \begin{array}{l}(1,13,8,19,3,15,10,21,5,17,12,23)\cdot \\ (2,14,9,20,4,16,11,22,6,18,7,24)\end{array}$$

As a simpler example, take $m = 3, r = 2$ *and* $\sigma = (1,2) \in S_2$. *Here we have*

$$A_1^3 = (1,2,3)$$
$$A_2^3 = (4,5,6)$$

note that since $m = 3$ *is prime then for any choice of* t_i, *the sum of the* t_i *would be either* $0(mod\ 3)$ *or coprime to* $m = 3$. *This means that there are only two possible cycle decompositions of* $C^{(1,2),3}_{t_1,t_2}$ *: It either consists of three transpositions (in which case* $l = 1, \frac{m}{l} = 3$ *and* $rl = 2$*) or it is a single* 6*-cycle (in which case* $l = 3, \frac{m}{l} = 1$ *and* $rl = 6$*). Let us exhibit the two possibilities. If we choose for example* $t_1 = 1, t_2 = 2$ *then in this case* $t_1 + t_2 \equiv 0(mod\ 3)$ *and we get*

$$C^{(1,2),3}_{1,2} = (1,6)(2,4)(3,5)$$

choosing $t_1 = 1, t_2 = 0$ *we get*

$$C^{(1,2),3}_{1,0} = (1,4,2,5,3,6)$$

As an example for the more general formula in Notation 3.6 let us take $m = 2, r = 2$ *and the* $k-1$ *transpositions* $(i, i+1)$ *for* $i = 1 \ldots k-1$ *for some* $k \in \mathbb{N}$. *We have*

$$A_i^2 = (2i-1, 2i)$$

As in the case of $m = 3$, *since* 2 *is prime there are only two possible cycle decompositions of* $C^{(i,i+1),2}_{t_1,t_2}$ *: Either a single* 4*-cycle or two transpositions. Let us give examples exhibiting both possibilities*

$$C^{(i,i+1),2}_{0,1} = (2i-1, 2i+2, 2i, 2i+1) \quad i = 1 \ldots k-1$$

$$C_{1,1}^{(i,i+1),2} = (2i-1, 2i+2)(2i, 2i+1) \quad i = 1 \ldots k-1$$

note that we can rewrite the definition of the model homomorphisms from Definition 3.1 in the following way

$$\varphi_1(\sigma_i) = C_{0,1}^{(i,i+1),2}$$

$$\varphi_2(\sigma_i) = A_1 \cdots A_{i-1} \cdot C_{0,0}^{(i,i+1),2} \cdot A_{i+2} \cdots A_k$$

$$\varphi_3(\sigma_i) = A_1 \cdots A_{i-1} \cdot C_{0,1}^{(i,i+1),2} \cdot A_{i+2} \cdots A_k$$

$\bullet$

We are now ready to present our main results. We first exhibit a family of homomorphisms $B_k \to S_{mk}$ for every $2 \leq m \in \mathbb{N}$ which is a natural generalization of Definition 3.1.

Let $m \in \mathbb{N}$, for each $6 < k \in \mathbb{N}$ we define

Definition 3.9. *The following homomorphisms $\psi, \phi \colon B_k \to \mathbf{S}_{mk}$ are called the **model** ones:*

$$\psi_m(\sigma_i) = C_{1,0}^{(i,i+1),m}$$

in other words, $\psi_m(\sigma_i)$ is a $2m$-cycle whose support is $supp(A_i^m) \cup supp(A_{i+1}^m) = \{1+m(i-1), 2+m(i-1), \ldots, m(i+1)\}$ and for each $0 \leq j \leq m-1$

$$\psi_m(\sigma_i)(a_j^i) = a_j^{i+1}$$
$$\psi_m(\sigma_i)(a_j^{i+1}) = a_{j+1}^i$$

where

$$a_j^i = 1 + m(i-1) + j \in supp(A_i^m) \quad j \in \bigtriangledown_m.$$

Still put differently, we can write

$$\psi_m(\sigma_i) = (a_0^i, a_0^{i+1}, a_1^i, a_1^{i+1}, \ldots, a_{m-1}^i, a_{m-1}^{i+1})$$

(note that $\psi_1 = \mu$ the canonical epimorphism, see Definition 2.7).

We now use ψ to define ϕ. For each divisor l of m where $1 \leq l < m$ and for each set of elements $\{t^i_{(i-1)l+1}, \dots, t^i_{(i+1)l}\}_{i=1,\dots,k-1}$ in $\bigtriangledown_{\frac{m}{l}}$

which satisfy the following condition

$$\forall j \in \bigtriangledown_l \quad t^i_{(i-1)l+1+|1+j|_l} + t^i_{il+1+|1+j|_l} \equiv t^{i+1}_{(i+1)l+1+j} + t^{i+1}_{il+1+|1+j|_l} (mod\ \frac{m}{l}) \tag{3.10}$$

define

$$\phi_{m,l,\{t^i_{(i-1)l+1},\dots,t^i_{(i+1)l}\}_{i=1,\dots,k-1}}(\sigma_i) = A_1^{\frac{m}{l}} \cdots A_{(i-1)l}^{\frac{m}{l}} \cdot C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\dots,t^i_{l(i+1)}} \cdot A_{(i+1)l+1}^{\frac{m}{l}} \cdots A_{kl}^{\frac{m}{l}} \tag{3.11}$$

Equation (3.10) *is a necessary and sufficient condition for* (3.11) *to be a homomorphism as we prove in Lemma 3.13.*

Note that in (3.11)*, since $\psi_l(\sigma_i)$ is a $2l$-cycle in $\mathbf{S}(l(i-1)+1,\dots,l(i+1))$, each permutation $C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\dots,t^i_{l(i+1)}}$ is considered as a permutation in $\mathbf{S}(\bigcup_{j=l(i-1)+1,\dots,l(i+1)} supp(A_j^{\frac{m}{l}}))$ (see also remark 2.5).*

In each of the above formulas $i = 1,...,k-1$. A homomorphism $\omega\colon B_k \to \mathbf{S}_{mk}$ is said to be **standard** *if it is conjugate to one of the model homomorphisms $\psi,\phi\colon B_k \to \mathbf{S}_{mk}$.*

•

Remark 3.10. *Note that by Remark 3.7, in the case $m = l$ in Definition 3.9, we have that all t_i are in $\bigtriangledown_1 = \{0\}$ and*

$$\phi_{m,m,\{t^i_{(i-1)l+1},\dots,t^i_{(i+1)l}\}_{i=1,\dots,k-1}}(\sigma_i) = C^{\psi_m(\sigma_i),1}_{t^i_{l(i-1)+1},\dots,t^i_{l(i+1)}} = C^{\psi_m(\sigma_i),1}_{0,\dots,0} = \psi_m(\sigma_i).$$

Hence ψ can be considered as a special case of ϕ.

We can also see from Example 3.8 that $\varphi_2 = \phi_{2,1,\{t^i_i=0,t^i_{i+1}=0\}}$ and $\varphi_3 = \phi_{2,1,\{t^i_i=0,t^i_{i+1}=1\}}$. Furthermore, as a consequence of our results in Section 4 we have that φ_1 is conjugate to ψ_2 and so we conclude that the model homomorphisms which we define in Definition 3.9 are indeed a generalization of the homomorphisms defined in Definition 3.1. •

We dedicate the rest of this subsection to prove that the maps defined in Definition 3.9 are indeed transitive and non-cyclic homomorphisms. We begin with the following

Lemma 3.11. *Let $m \in \mathbb{N}$. The map $\psi_m\colon B_k \to S_{mk}$ defined as follows (see also Definition 3.9)*

$$\psi_m(\sigma_i) = (a^i_0, a^{i+1}_0, a^i_1, a^{i+1}_1, \dots, a^i_{m-1}, a^{i+1}_{m-1}), \quad 1 \leq i \leq k-1$$

where $a^i_j = 1 + m(i-1) + j \in supp(A^m_i)$ *and* $j \in \bigtriangledown_m$*, is a well defined non-cyclic transitive homomorphism.*

Proof. For each $j \in \bigtriangledown_m$ and $1 \le i \le k-1$ we have

$$\begin{aligned}&\psi_m(\sigma_i)\psi_m(\sigma_{i+1})\psi_m(\sigma_i)(a^i_j) =\\ &\psi_m(\sigma_i)\psi_m(\sigma_{i+1})(a^{i+1}_j) =\\ &\psi_m(\sigma_i)(a^{i+2}_j) = a^{i+2}_j\end{aligned}$$

$$\begin{aligned}&\psi_m(\sigma_{i+1})\psi_m(\sigma_i)\psi_m(\sigma_{i+1})(a^i_j) =\\ &\psi_m(\sigma_{i+1})\psi_m(\sigma_i)(a^i_j) =\\ &\psi_m(\sigma_{i+1})(a^{i+1}_j) = a^{i+2}_j\end{aligned}$$

$$\begin{aligned}&\psi_m(\sigma_i)\psi_m(\sigma_{i+1})\psi_m(\sigma_i)(a^{i+1}_j) =\\ &\psi_m(\sigma_i)\psi_m(\sigma_{i+1})(a^i_{j+1}) =\\ &\psi_m(\sigma_i)(a^i_{j+1}) = a^{i+1}_{j+1}\end{aligned}$$

$$\begin{aligned}&\psi_m(\sigma_{i+1})\psi_m(\sigma_i)\psi_m(\sigma_{i+1})(a^{i+1}_j) =\\ &\psi_m(\sigma_{i+1})\psi_m(\sigma_i)(a^{i+2}_j) =\\ &\psi_m(\sigma_{i+1})(a^{i+2}_j) = a^{i+1}_{j+1}\end{aligned}$$

$$\begin{aligned}&\psi_m(\sigma_i)\psi_m(\sigma_{i+1})\psi_m(\sigma_i)(a^{i+2}_j) =\\ &\psi_m(\sigma_i)\psi_m(\sigma_{i+1})(a^{i+2}_j) =\\ &\psi_m(\sigma_i)(a^{i+1}_{j+1}) = a^i_{j+2}\end{aligned}$$

$$\begin{aligned}&\psi_m(\sigma_{i+1})\psi_m(\sigma_i)\psi_m(\sigma_{i+1})(a^{i+2}_j) =\\ &\psi_m(\sigma_{i+1})\psi_m(\sigma_i)(a^{i+1}_{j+1}) =\\ &\psi_m(\sigma_{i+1})(a^i_{j+2}) = a^i_{j+2}\end{aligned}$$

Hence, since $a^q_j \notin supp(\psi_m(\sigma_i)) \cup supp(\psi_m(\sigma_{i+1}))$ for $q \neq i, i+1, i+2$ and $j \in \bigtriangledown_m$, we have that for each $1 \le i \le k-1$

$$\psi_m(\sigma_i)\psi_m(\sigma_{i+1})\psi_m(\sigma_i) = \psi_m(\sigma_{i+1})\psi_m(\sigma_i)\psi_m(\sigma_{i+1}).$$

Furthermore, we have that $\psi_m(\sigma_i)\psi_m(\sigma_j) = \psi_m(\sigma_j)\psi_m(\sigma_i)$ for $|i-j| > 1$ since for these indices $supp(\psi_m(\sigma_i)) \cap supp(\psi_m(\sigma_j)) = (supp(A_i) \cup supp(A_{i+1})) \cap (supp(A_j) \cup supp(A_{j+1})) = \emptyset$.

Hence we conclude that $\psi_m \colon B_k \to S_{mk}$ is indeed a homomorphism. It is clear that ψ_m is non-cyclic since $\psi_m(\sigma_i) \neq \psi_m(\sigma_j)$ for $i \neq j$. To see that ψ_m is transitive, consider the orbit of $1 \in \Delta_{mk}$ in $Im(\psi_m)$; it includes $supp(\psi_m(\sigma_1)) = supp(A_1) \cup supp(A_2)$ and since $supp(\psi_m(\sigma_i)) \cap supp(\psi_m(\sigma_{i+1})) = supp(A_{i+1})$ then it is easy to see by induction on i that the orbit of 1 includes $\cup_{i=1,\dots,k} supp(A_i) = \Delta_{mk}$.

□

Before proceeding to show that the rest of the maps we defined in Definition 3.9 are well defined non-cyclic and transitive homomorphisms, we prove a few preliminary results

Lemma 3.12. *Let $k, l \in \mathbb{N}$ and $A_1, \dots, A_{kl}, C_1, \dots, C_{k-1}$ be permutations in S_n for some $n \in \mathbb{N}$ which satisfy the following restrictions for each $i = 1, \dots, k-1$*

$$A_r^{C_i} = \begin{cases} A_r & r = 1, \dots, (i-1)l \quad or \quad r = (i+1)l+1, \dots, kl \\ A_{\psi_l(\sigma_i)(r)} & r = (i-1)l+1, \dots, (i+1)l \end{cases} \tag{3.12}$$

$$[A_r, A_s] = 1 \quad r, s = 1, \dots, kl \tag{3.13}$$

where ψ_l is the model homomorphism from Definition 3.9, then we have that $C_i \infty C_{i+1}$ (i.e. C_i and C_{i+1} are a braid-like pair) if and only if $D_i \infty D_{i+1}$ where

$$D_i = A_1 \cdots A_{(i-1)l} \cdot C_i \cdot A_{(i+1)l+1} \cdots A_{kl}$$

for each $i = 1, \dots, k-1$

Proof. First note that since

$$\psi_l(\sigma_i)(\{(i-1)l+1, \dots, (i+1)l\}) = \{(i-1)l+1, \dots, (i+1)l\}$$

we have that the conditions 3.12 and 3.13 imply that

$$(A_{(i-1)l+1} \cdots A_{(i+1)l})^{C_i} = A_{(i-1)l+1} \cdots A_{(i+1)l}$$

and so we can deduce the following two equations

$$C_i^{A_{(i-1)l+1} \cdots A_{il} \cdot A_{il+1} \cdots A_{(i+1)l}} = C_i \tag{3.14}$$

$$C_{i+1}^{A_{il+1} \cdots A_{(i+1)l} \cdot A_{(i+1)l+1} \cdots A_{(i+2)l}} = C_{i+1} \tag{3.15}$$

Now, by definition, $D_i \infty D_{i+1}$ means that

$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$$

let us write the last equation as follows

$$D_i^{D_{i+1}} = D_{i+1}^{D_i^{-1}}$$

and explicitly

$$\begin{aligned}
&(A_1 \cdots A_{(i-1)l} \cdot C_i \cdot A_{(i+1)l+1} \cdots A_{kl})^{A_1 \cdots A_{il} \cdot C_{i+1} \cdot A_{(i+2)l+1} \cdots A_{kl}} = \\
&(A_1 \cdots A_{il} \cdot C_{i+1} \cdot A_{(i+2)l+1} \cdots A_{kl})^{A_{kl}^{-1} \cdots A_{(i+1)l+1}^{-1} \cdot C_i^{-1} \cdot A_{(i-1)l}^{-1} \cdots A_1^{-1}}.
\end{aligned} \tag{3.16}$$

Now since $D_i = A_1 \cdots A_{(i-1)l} \cdot C_i \cdot A_{(i+1)l+1} \cdots A_{kl}$ and the cycles A_j in this product are disjoint and C_i commutes with all of these A_j (according to (3.12)) we conclude that all the cycles in this product commute and we can write

$$D_i^{-1} = A_{kl}^{-1} \cdots A_{(i+1)l+1}^{-1} \cdot C_i^{-1} \cdot A_{(i-1)l}^{-1} \cdots A_1^{-1} =$$

$$= A_1^{-1} \cdots A_{(i-1)l}^{-1} \cdot C_i^{-1} \cdot A_{(i+1)l+1}^{-1} \cdots A_{kl}^{-1}.$$

Hence we can rewrite (3.16) as

$$\begin{aligned}
&(A_1 \cdots A_{(i-1)l} \cdot C_i \cdot A_{(i+1)l+1} \cdots A_{kl})^{A_1 \cdots A_{il} \cdot C_{i+1} \cdot A_{(i+2)l+1} \cdots A_{kl}} = \\
&(A_1 \cdots A_{il} \cdot C_{i+1} \cdot A_{(i+2)l+1} \cdots A_{kl})^{A_1^{-1} \cdots A_{(i-1)l}^{-1} \cdot C_i^{-1} \cdot A_{(i+1)l+1}^{-1} \cdots A_{kl}^{-1}}.
\end{aligned}$$

Now using the given equations 3.12 and 3.13 we can rewrite the last equation as follows (we denote $\psi_l(\sigma_i)$ as $\widehat{\sigma}_i$ for short)

$$\begin{aligned}
&A_1 \cdots A_{(i-1)l} C_i^{A_{(i-1)l+1} \cdots A_{il} \cdot C_{i+1} \cdot A_{(i+2)l+1} \cdots A_{kl}} \cdot \\
&\cdot A_{\widehat{\sigma}_{i+1}((i+1)l+1)} \cdots A_{\widehat{\sigma}_{i+1}((i+2)l)} A_{(i+2)l+1} \cdots A_{kl} =
\end{aligned}$$

$$A_1 \cdots A_{(i-1)l} A_{\widehat{\sigma}_i^{-1}((i-1)l+1)} \cdots A_{\widehat{\sigma}_i^{-1}(il)} C_{i+1}^{C_i^{-1} \cdot A_{(i+1)l+1}^{-1} \cdots A_{kl}^{-1}} A_{(i+2)l+1} \cdots A_{kl}$$

which can be written after cancellations as

$$C_i^{A_{(i-1)l+1}\cdots A_{il}\cdot C_{i+1}\cdot A_{(i+2)l+1}\cdots A_{kl}} A_{\widehat{\sigma}_{i+1}((i+1)l+1)}\cdots A_{\widehat{\sigma}_{i+1}((i+2)l)} = \\ A_{\widehat{\sigma}_i^{-1}((i-1)l+1)}\cdots A_{\widehat{\sigma}_i^{-1}(il)} C_{i+1}^{C_i^{-1}\cdot A^{-1}_{(i+1)l+1}\cdots A^{-1}_{kl}} \tag{3.17}$$

now since

$$\widehat{\sigma}_i^{-1}(\{(i-1)l+1,\ldots,il\}) = \{il+1,\ldots,(i+1)l\} = \\ = \widehat{\sigma}_{i+1}(\{(i+1)l+1,\ldots,(i+2)l\})$$

and by the condition 3.13, we have that

$$A_{\widehat{\sigma}_{i+1}((i+1)l+1)}\cdots A_{\widehat{\sigma}_{i+1}((i+2)l)} = A_{il+1}\cdots A_{(i+1)l}$$

and

$$A_{\widehat{\sigma}_i^{-1}((i-1)l+1)}\cdots A_{\widehat{\sigma}_i^{-1}(il)} = A_{il+1}\cdots A_{(i+1)l}$$

and so we can rewrite (3.17) as

$$C_i^{A_{(i-1)l+1}\cdots A_{il}\cdot C_{i+1}\cdot A_{(i+2)l+1}\cdots A_{kl}} A_{il+1}\cdots A_{(i+1)l} = \\ A_{il+1}\cdots A_{(i+1)l} C_{i+1}^{C_i^{-1}\cdot A^{-1}_{(i+1)l+1}\cdots A^{-1}_{kl}} \tag{3.18}$$

or equivalently as

$$C_i^{A_{(i-1)l+1}\cdots A_{il}\cdot C_{i+1}\cdot A_{(i+2)l+1}\cdots A_{kl} A_{il+1}\cdots A_{(i+1)l}} = \\ = C_{i+1}^{C_i^{-1}\cdot A^{-1}_{(i+1)l+1}\cdots A^{-1}_{(i+2)l}}$$

which is equivalent to

$$C_i^{A_{(i-1)l+1}\cdots A_{il}\cdot C_{i+1}\cdot A_{(i+2)l+1}\cdots A_{kl} A_{il+1}\cdots A_{(i+1)l}\cdot A_{(i+2)l}\cdots A_{(i+1)l+1}\cdot C_i} = C_{i+1}$$

which can be rewritten, using 3.12 and 3.13, as

$$C_i^{A_{(i-1)l+1}\cdots A_{il}\cdot C_{i+1}\cdot A_{il+1}\cdots A_{(i+1)l} A_{(i+2)l+1}\cdots A_{kl}\cdot C_i A_{(i+1)l+1}\cdots A_{(i+2)l}} = C_{i+1}$$

now according to (3.14), we can rewrite the last equation as

$$C_i^{(A_{il+1}\cdots A_{(i+1)l})^{-1}\cdot C_{i+1}\cdot (A_{il+1}\cdots A_{(i+1)l})A_{(i+2)l+1}\cdots A_{kl}\cdot C_i A_{(i+1)l+1}\cdots A_{(i+2)l}} = C_{i+1}$$

and according to (3.15) we can rewrite the last equation as

$$C_i^{(A_{(i+1)l+1}\cdots A_{(i+2)l})\cdot C_{i+1}\cdot (A_{(i+1)l+1}\cdots A_{(i+2)l})^{-1} A_{(i+2)l+1}\cdots A_{kl}\cdot C_i A_{(i+1)l+1}\cdots A_{(i+2)l}} = C_{i+1}$$

and using 3.12 and 3.13 again we get that this is equivalent to

$$C_i^{C_{i+1}C_i} = C_{i+1}$$

□

Lemma 3.13. *Let $l \in \mathbb{N}$, $\psi_l \colon B_k \to S_{kl}$ be the model homomorphism defined for each $i = 1, \ldots, k-1$ as follows (see Definition 3.9 and Notations 3.4 and 3.6)*

$$\psi_l(\sigma_i) = (a_0^i, a_0^{i+1}, a_1^i, a_1^{i+1}, \ldots, a_{l-1}^i, a_{l-1}^{i+1})$$

where

$$a_j^i = 1 + l(i-1) + j \in supp(A_i^l) \quad j \in \bigtriangledown_l$$

Let us denote

$$\Delta^i = \bigcup_{j=0,\ldots,l-1} supp(A_{(i-1)l+1+j}). \tag{3.19}$$

Then for each $i = 1, \ldots, k-2$, $n \in \mathbb{N}$ and $t^i_{l(i-1)+1}, \ldots, t^i_{l(i+1)}, t^{i+1}_{li+1}, \ldots, t^{i+1}_{l(i+2)} \in \bigtriangledown_n$ we have that

$$C^{\psi_l(\sigma_i),n}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}} \infty C^{\psi_l(\sigma_{i+1}),n}_{t^{i+1}_{li+1},\ldots,t^{i+1}_{l(i+2)}}$$

if and only if

$$(C^{\psi_l(\sigma_i),n}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}})^2\big|_{\Delta^{i+1}} = (C^{\psi_l(\sigma_{i+1}),n}_{t^{i+1}_{li+1},\ldots,t^{i+1}_{l(i+2)}})^2\big|_{\Delta^{i+1}}$$

if and only if

$$\forall j \in \bigtriangledown_l \quad t^i_{(i-1)l+1+|1+j|_l} + t^i_{il+1+|1+j|_l} \equiv t^{i+1}_{(i+1)l+1+j} + t^{i+1}_{il+1+|1+j|_l} (mod\ n)$$

(The n in this lemma is $\frac{m}{l}$ appearing in (3.10) *in Definition 3.9)*

Proof. Let us denote $C^{\psi_l(\sigma_i),n}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}} \in S_{lnk}$ as C_i for short.

Recall that we consider each element of S_{lnk} as a bijection from the following set to itself

$$\Delta_{lnk} = \{1, \ldots, lnk\} = \bigcup_{1,\ldots,lk} supp(A_i)$$

We denote

$$\Delta^i = \bigcup_{j=0,\ldots,l-1} supp(A_{(i-1)l+1+j})$$

Using the notation in (3.19), we can write

$$supp(C_i) = \Delta^i \cup \Delta^{i+1}$$

and so

$$C_i|_{\Delta_{lnk}-(\Delta^i\cup\Delta^{i+1})} = id \tag{3.20}$$

and

$$C_{i+1}|_{\Delta_{lnk}-(\Delta^{i+1}\cup\Delta^{i+2})} = id \tag{3.21}$$

we can also see that C_i maps Δ^i bijectively to Δ^{i+1} and vice versa, i.e.

$$C_i \colon \Delta^i \xrightarrow{\cong} \Delta^{i+1} \xrightarrow{\cong} \Delta^i \tag{3.22}$$

and

$$C_{i+1}\colon \Delta^{i+1} \xrightarrow{\cong} \Delta^{i+2} \xrightarrow{\cong} \Delta^{i+1} \tag{3.23}$$

Consider the following permutation in S_{lnk}

$$D_i = C_i C_{i+1} C_i C_{i+1}^{-1} C_i^{-1} C_{i+1}^{-1}$$

We shall show first that D_i is trivial (as a permutation in S_{lnk}) iff $C_i^2 = C_{i+1}^2$. According to equations 3.20 and 3.21 it is enough to show this for the restriction of D_i to $\Delta^i \cup \Delta^{i+1} \cup \Delta^{i+2}$.

Consider first the following restriction

$$D_i|_{\Delta^i} = C_i C_{i+1} C_i C_{i+1}^{-1} C_i^{-1} C_{i+1}^{-1}|_{\Delta^i}$$

because of equation 3.21 we have

$$C_i C_{i+1} C_i C_{i+1}^{-1} C_i^{-1} C_{i+1}^{-1}|_{\Delta^i} = C_i C_{i+1} C_i C_{i+1}^{-1} C_i^{-1}|_{\Delta^i}$$

and by the diagrams 3.22 and 3.23 we have that

$$C_{i+1}^{-1} C_i^{-1}(\Delta^i) = \Delta^{i+2}$$

and by equation 3.20 we have that $C_i^{-1}|_{\Delta^{i+2}} = id$ and so

$$C_i C_{i+1} C_i (C_{i+1}^{-1} C_i^{-1}|_{\Delta^i}) = C_i C_{i+1} C_{i+1}^{-1} C_i^{-1}|_{\Delta^i} = id$$

hence we conclude that $D_i|_{\Delta^i} = id$.

Now consider

$$D_i|_{\Delta^{i+2}} = C_i C_{i+1} C_i C_{i+1}^{-1} C_i^{-1} C_{i+1}^{-1}|_{\Delta^{i+2}}$$

by the diagrams 3.22 and 3.23 we have that

$$C_i^{-1} C_{i+1}^{-1}(\Delta^{i+2}) = \Delta^i$$

by equation 3.21 we have that $C_{i+1}|_{\Delta^i} = id$ and so

$$C_iC_{i+1}C_iC_{i+1}^{-1}(C_i^{-1}C_{i+1}^{-1}|_{\Delta^{i+2}}) = C_iC_{i+1}C_iC_i^{-1}C_{i+1}^{-1}|_{\Delta^{i+2}} = C_i|_{\Delta^{i+2}} = id$$

the last equation implied from equation 3.20. Hence we conclude that $D_i|_{\Delta^{i+2}} = id$.

Now consider

$$D_i|_{\Delta^{i+1}} = C_iC_{i+1}C_iC_{i+1}^{-1}C_i^{-1}C_{i+1}^{-1}|_{\Delta^{i+1}}$$

by the diagram 3.23 we have that

$$C_{i+1}^{-1}(\Delta^{i+1}) = \Delta^{i+2}$$

and by equation 3.20 we have that $C_i^{-1}|_{\Delta^{i+2}} = id$ and so

$$C_iC_{i+1}C_iC_{i+1}^{-1}C_i^{-1}(C_{i+1}^{-1}|_{\Delta^{i+1}}) = C_iC_{i+1}C_iC_{i+1}^{-1}C_{i+1}^{-1}|_{\Delta^{i+1}}$$

now by the diagrams 3.22 and 3.23 we have that

$$C_iC_{i+1}^{-1}C_{i+1}^{-1}(\Delta^{i+1}) = \Delta^i$$

and by equation 3.21 we have that $C_{i+1}|_{\Delta^i} = id$ and so

$$C_iC_{i+1}(C_iC_{i+1}^{-1}C_{i+1}^{-1}|_{\Delta^{i+1}}) = C_iC_iC_{i+1}^{-1}C_{i+1}^{-1}|_{\Delta^{i+1}}$$

Hence we conclude that $supp(D_i) = \Delta^{i+1}$ and that $D_i|_{\Delta^{i+1}} = C_i^2C_{i+1}^{-2}|_{\Delta^{i+1}}$.

Since $D_i = C_iC_{i+1}C_iC_{i+1}^{-1}C_i^{-1}C_{i+1}^{-1} = id$ iff $C_i \infty C_{i+1}$, we have that

$$C_i \infty C_{i+1} \quad \text{iff} \quad C_i^2|_{\Delta^{i+1}} = C_{i+1}^2|_{\Delta^{i+1}}$$

Now let us see what this means in terms of the indices t_*. Let us write explicitly

$$(C^{\psi_l(\sigma_i),n}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}})^2\big|_{\Delta^{i+1}} = (C^{\psi_l(\sigma_{i+1}),n}_{t^{i+1}_{li+1},\ldots,t^{i+1}_{l(i+2)}})^2\big|_{\Delta^{i+1}} \tag{3.24}$$

Let us denote $\psi_l(\sigma_i)$ as $\widehat{\sigma}_i$ for short. We have by definition that

$$C_i(a^r_s) = a^{\widehat{\sigma}_i(r)}_{|s+t^i_{\widehat{\sigma}_i(r)}|_n}$$

where (see notation 3.4)

$$a^r_s = 1 + n(r-1) + s \in supp(A^n_r), \quad s \in \bigtriangledown_n$$

since we have to check equation 3.24 only on the set Δ^{i+1} we have that it is equivalent to

$$a^{\widehat{\sigma}^2_i(r)}_{|s+t^i_{\widehat{\sigma}_i(r)}+t^i_{\widehat{\sigma}^2_i(r)}|_n} = a^{\widehat{\sigma}^2_{i+1}(r)}_{|s+t^{i+1}_{\widehat{\sigma}_{i+1}(r)}+t^{i+1}_{\widehat{\sigma}^2_{i+1}(r)}|_n}$$

for $r = il+1, \dots, (i+1)l$ and each $s \in \bigtriangledown_n$. But for each $j \in \bigtriangledown_l$ we have

$$r = il + 1 + j \xrightarrow{\widehat{\sigma}_i} (i-1)l + 1 + |1+j|_l \xrightarrow{\widehat{\sigma}_i} il + 1 + |1+j|_l$$

$$r = il + 1 + j \xrightarrow{\widehat{\sigma}_{i+1}} (i+1)l + 1 + j \xrightarrow{\widehat{\sigma}_{i+1}} il + 1 + |1+j|_l$$

hence we conclude that for each $r = il+1, \dots, (i+1)l$ we have that $\widehat{\sigma}^2_i(r) = \widehat{\sigma}^2_{i+1}(r)$ and that equation 3.24 holds iff $t^i_{\widehat{\sigma}_i(r)} + t^i_{\widehat{\sigma}^2_i(r)} \equiv t^{i+1}_{\widehat{\sigma}_{i+1}(r)} + t^{i+1}_{\widehat{\sigma}^2_{i+1}(r)} (\mathrm{mod}\ n)$ for each $r = il+1, \dots, (i+1)l$ which translates to

$$\forall j \in \bigtriangledown_l \quad t^i_{(i-1)l+1+|1+j|_l} + t^i_{il+1+|1+j|_l} \equiv t^{i+1}_{(i+1)l+1+j} + t^{i+1}_{il+1+|1+j|_l} (\mathrm{mod}\ n)$$

for all $i = 1, \dots, k-2$.

□

We are now ready to prove

Proposition 3.14. *Let $m \in \mathbb{N}$. The model homomorphisms $\psi, \phi\colon B_k \to S_{mk}$ (see Definition 3.9) are well-defined non-cyclic transitive homomorphisms*

Proof. That ψ_m is a well defined non-cyclic transitive homomorphism is proved in Lemma 3.11.

Now let l be a divisor of m where $1 \leq l < m$ and let

$$\{t^i_{(i-1)l+1}, \dots, t^i_{(i+1)l}\}_{i=1,\dots,k-1} \in \bigtriangledown_{\frac{m}{l}}$$

(when $l = 1$ we take $\bigtriangledown_1 = \{0\}$)

then for each $i = 1, \ldots, k-1$ we define

$$\phi_{m,l,\{t^i_{(i-1)l+1},\ldots,t^i_{(i+1)l}\}_{i=1,\ldots,k-1}}(\sigma_i) = A_1^{\frac{m}{l}} \cdots A_{(i-1)l}^{\frac{m}{l}} \cdot C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}} \cdot A_{(i+1)l+1}^{\frac{m}{l}} \cdots A_{kl}^{\frac{m}{l}}$$

Let us denote $A_*^{\frac{m}{l}}$ as A_*, $C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}}$ as C_i and $\phi_{m,l,\{t^i_{(i-1)l+1},\ldots,t^i_{(i+1)l}\}_{i=1,\ldots,k-1}}(\sigma_i)$ as $\widehat{\sigma}_i$ for short. We have by definition of C_i that

$$A_r^{C_i} = \begin{cases} A_r & r = 1, \ldots, (i-1)l \quad \text{or} \quad r = (i+1)l+1, \ldots, kl \\ A_{\psi_l(\sigma_i)(r)} & r = (i-1)l+1, \ldots, (i+1)l \end{cases} \tag{3.25}$$

and clearly

$$[A_r, A_s] = 1 \quad r, s = 1, \ldots, kl \tag{3.26}$$

According to Lemma 3.12 we have that $\widehat{\sigma}_i \infty \widehat{\sigma}_{i+1}$ iff $C_i \infty C_{i+1}$ iff

$$\forall j \in \bigtriangledown_l \quad t^i_{(i-1)l+1+|1+j|_l} + t^i_{il+1+|1+j|_l} \equiv t^{i+1}_{(i+1)l+1+j} + t^{i+1}_{il+1+|1+j|_l} (\text{mod } \frac{m}{l})$$

which is exactly the condition stated in the definition of ϕ (see Definition 3.9). Now suppose $|i-j| \geq 2$ where $1 \leq i < j \leq k-1$. Then since $j - i \geq 2$ we have that $i+1 \leq j-1$ and we can write

$$\widehat{\sigma}_j = A_1 \cdots A_{(i-1)l}(A_{(i-1)l+1} \cdots A_{(i+1)l})A_{(i+1)l+1} \cdots A_{(j-1)l} C_j A_{(j+1)l+1} \cdots A_{kl}$$

Now $[C_i, C_j] = 1$ since $supp(C_i) \cap supp(C_j) = \emptyset$, $[C_i, A_r] = 1$ for $r = 1, \ldots, (i-1)l$ and $r = (i+1)l+1, \ldots, kl$ according to equation 3.25, and finally we have that the equations 3.25 and 3.26 imply (see the proof of Lemma 3.12)

$$[C_i, A_{(i-1)l+1} \cdots A_{(i+1)l}] = 1$$

and this means that $[\widehat{\sigma}_i, \widehat{\sigma}_j] = 1$ for $|i-j| > 1$. Hence we conclude that ϕ is indeed a homomorphism.

ϕ is non-cyclic since $\widehat{\sigma}_i \neq \widehat{\sigma}_{i+1}$ for $i \neq j$.

To see that ϕ is transitive, consider the orbit of $1 \in \Delta_{mk}$ in $Im(\phi)$. We see from $\widehat{\sigma}_2$ that $1 \in supp(A_1)$ hence $supp(A_1) \subseteq orbit(1)$. Now since the support of each cycle in the cycle decomposition of C_1 contains an element of $supp(A_1)$ we have, by considering $\widehat{\sigma}_1$, that $supp(C_1) \subseteq orbit(1)$. Now since $supp(C_i) \cap supp(C_{i+1}) = supp(A_{il+1}) \cup \cdots \cup supp(A_{(i+1)l})$ and the support of each cycle in the cycle decomposition of C_{i+1} contains an element of $supp(A_{il+1}) \cup \cdots \cup supp(A_{(i+1)l})$ for each $i = 1, \ldots, k-1$, we can see (by induction on i) that

$$\bigcup_{i=1,\ldots,k-1} supp(C_i) \subseteq orbit(1)$$

but since $supp(C_i) = supp(A_{(i-1)l+1}) \cup \cdots \cup supp(A_{(i+1)l})$ this means that

$$\bigcup_{i=1,\ldots,k-1} supp(A_{(i-1)l+1}) \cup \cdots \cup supp(A_{(i+1)l}) =$$

$$= \bigcup_{i=1,\ldots,kl} A_i = \Delta_{mk} \subseteq orbit(1)$$

and hence ϕ is transitive.

□

3.3 Good Permutation Representations

In this subsection we define and study a particular property of permutation representations of the braid group. This property is common to almost all known permutation representations of the braid group (except for a finite number of exceptions, i.e., homomorphisms $B_k \to S_{mk}$ for small values of k and m - see section 4 in [Lin04b] for these examples). We define this property in the next definition and explain the motivation for studying it the remark that follows.

Definition 3.15. *We call a homomorphism $\omega \colon B_k \to S_n$ **good** if one of the following holds*

1. $supp(\widehat{\sigma}_i) \cap supp(\widehat{\sigma}_j) = \emptyset$ *for* $1 \leq i, j \leq k-1$ *such that* $|i-j| > 1$.

2. $supp(\widehat{\sigma}_1) = \cdots = supp(\widehat{\sigma}_{k-1})$

•

Remark 3.16. *The motivation for defining good homomorphisms in Definition 3.15 is our conjecture that in fact all transitive non-cyclic permutation representations of the braid group are good (see Conjecture 6.1). Furthermore, it is our conjecture that all good transitive non-cyclic permutations representations of the braid group are either standard or derived in some*

sense from standard homomorphisms. We make these conjectures precise in Section 6. Hence, we dedicate the rest of the section to the investigation of good permutation representations of the braid group. •

Lemma 3.17. *Let $\omega, \omega' \colon B_k \to S_n$ be conjugate homomorphisms, then*

$$|supp(\omega(\sigma_1)) \cap supp(\omega(\sigma_2))| = |supp(\omega'(\sigma_1)) \cap supp(\omega'(\sigma_2))|$$

Proof. Suppose $\omega' = \omega^\zeta$ for some $\zeta \in S_n$. Let $j \in supp(\omega(\sigma_1)) \cap supp(\omega(\sigma_2))$ then

$$\zeta(j) \in supp(\omega^\zeta(\sigma_1)) \cap supp(\omega^\zeta(\sigma_2)) =$$

$$= supp(\omega'(\sigma_1)) \cap supp(\omega'(\sigma_2))$$

and similarly, if $\zeta(j) \in supp(\omega'(\sigma_1)) \cap supp(\omega'(\sigma_2))$ then by applying ζ^{-1} on $\zeta(j)$ we get that

$$j \in supp(\omega(\sigma_1)) \cap supp(\omega(\sigma_2))$$

Hence ζ is a bijection between the sets $supp(\omega(\sigma_1)) \cap supp(\omega(\sigma_2))$ and $supp(\omega'(\sigma_1)) \cap supp(\omega'(\sigma_2))$.

□

Definition 3.18. *For a homomorphism $\omega \colon B_k \to S_n$ define*

$$\mathbf{intersect}(\omega) = |\mathbf{supp}(\widehat{\sigma}_{\mathbf{1}}) \cap \mathbf{supp}(\widehat{\sigma}_{\mathbf{2}})|$$

and

$$\mathbf{supp}(\omega) = |\mathbf{supp}(\widehat{\sigma}_{\mathbf{1}})|$$

•

Remark 3.19. *Lemma 3.17 shows that for any homomorphism $\omega \colon B_k \to S_n$, $intersect(\omega)$ is invariant under conjugation. In particular, since $\alpha\sigma_i\alpha^{-1} = \sigma_{i+1}$ (where as above $\alpha = \sigma_1\sigma_2\cdots\sigma_{k-1}$) then by conjugating ω with $\omega(\alpha)$ we have that for any i (where $1 \le i \le k-2$)*

$$intersect(\omega) = |supp(\widehat{\sigma}_i) \cap supp(\widehat{\sigma}_{i+1})|$$

Similarly, since $x \mapsto \alpha(x)$ is a bijection between $supp(\widehat{\sigma}_1)$ and $supp(\widehat{\sigma}_1^\zeta)$ for any $\zeta \in S_n$, we see that $supp(\omega)$ is also invariant under conjugation and in particular we get that

$$supp(\omega) = |supp(\widehat{\sigma}_i)|$$

for each $i = 1, \dots, k-1$

$\bullet$

We also prove

Lemma 3.20. *Let* $\omega\colon B_k \to S_n$ *be a homomorphism and let* $x_1 \in supp(\widehat{\sigma}_i)$ *for some* i, $1 \le i \le k-1$. *If* $x_1 \notin supp(\widehat{\sigma}_{i\pm1})$ *then* $\widehat{\sigma}_i(x_1) \in supp(\widehat{\sigma}_{i\pm1})$ *(for* $i = 1$ *and* $i = k-1$ *we take* $\widehat{\sigma}_{1\pm1}$ *and* $\widehat{\sigma}_{(k-1)\pm1}$ *to mean* $\widehat{\sigma}_2$ *and* $\widehat{\sigma}_{k-2}$ *respectively)*

Proof. Let us denote $x_2 = \widehat{\sigma}_i(x_1)$. Assume by way of contradiction that $x_2 \notin supp(\widehat{\sigma}_{i\pm1})$. Let us write

$$D = (x_1, x_2, \dots)$$

where D is a cycle in the cycle decomposition of $\widehat{\sigma}_i$. Now since $\widehat{\sigma}_i = \widehat{\sigma}_{i\pm1}^{\widehat{\sigma}_i\widehat{\sigma}_{i\pm1}}$, we have that $\widehat{\sigma}_i^{\widehat{\sigma}_{i\pm1}^{-1}\widehat{\sigma}_i^{-1}} = \widehat{\sigma}_{i\pm1}$ and we expect that

$$D^{\widehat{\sigma}_{i\pm1}^{-1}\widehat{\sigma}_i^{-1}} = (\widehat{\sigma}_i\widehat{\sigma}_{i\pm1}(x_1), \widehat{\sigma}_i\widehat{\sigma}_{i\pm1}(x_2), \dots) = (x_2, \widehat{\sigma}_i(x_2), \dots)$$

be a cycle in the cycle decomposition of $\widehat{\sigma}_{i\pm1}$, i.e., that $x_2 \in supp(\widehat{\sigma}_{i\pm1})$ which is contrary to our assumption.

$\square$

As a consequence we get

Corollary 3.21. *Let* $\omega\colon B_k \to S_n$ *be a homomorphism. Then for every* $i = 1, \dots, k-1$ *and for every cycle* D *in the cycle decomposition of* $\widehat{\sigma}_i$ *we have that*

$$|supp(D) \cap supp(\widehat{\sigma}_{i\pm1})| \ge \frac{1}{2}|supp(D)|$$

and so

$$intersect(\omega) \ge \frac{1}{2} supp(\omega)$$

Proof. Let $D = (x_0, \dots, x_{l-1})$ be a cycle in the cycle decomposition of $\widehat{\sigma}_i$. Consider $(x_0, \dots, x_{l-1})$ as an ordered tuple of elements in Δ_n, then according to Lemma 3.20 for each $j = 0, \dots, l-1$ we have that at least one of x_j and $x_{|j+1|_l}$ are in $supp(\widehat{\sigma}_{i\pm1})$ which means that $|supp(D) \cap supp(\widehat{\sigma}_{i\pm1})| \geq \frac{1}{2}|supp(D)|$. Since this holds for every cycle in the cycle decomposition of $\widehat{\sigma}_1$, say $\widehat{\sigma}_1 = D_1 \cdots D_r$, we conclude that

$$intersect(\omega) = \sum_{j=1,\dots,r} |supp(D_j) \cap supp(\widehat{\sigma}_{i\pm1})| \geq$$

$$\geq \sum_{j=1,\dots,r} \frac{1}{2}|supp(D_j)| = \frac{1}{2}supp(\omega)$$

□

Let us now focus on good transitive homomorphisms $\omega\colon B_k \to S_{mk}$. We prove

Lemma 3.22. *Let $\omega\colon B_k \to S_{mk}$ be a good transitive homomorphism where $k \geq 4$, then exactly one of the following holds :*

1. $supp(\omega) = 2m \quad$ *and* $\quad intersect(\omega) = m$

2. $supp(\omega) = mk \quad$ *and* $\quad intersect(\omega) = mk$

Proof. Suppose $supp(\widehat{\sigma}_1) = \cdots = supp(\widehat{\sigma}_{k-1})$, then since ω is transitive we necessarily have that $supp(\widehat{\sigma}_1) = \cdots = supp(\widehat{\sigma}_{k-1}) = \Delta_{mk}$ but then also $supp(\widehat{\sigma}_i) \cap supp(\widehat{\sigma}_{i+1}) = \Delta_{mk}$ for $i = 1, \dots, k-2$ and so $supp(\omega) = mk$ and $intersect(\omega) = mk$.

Assume then that $supp(\widehat{\sigma}_i) \cap supp(\widehat{\sigma}_j) = \emptyset$ for $1 \leq i, j \leq k-1$ such that $|i-j| > 1$ and we will prove that $supp(\omega) = 2m$ and $intersect(\omega) = m$.

Since ω is transitive we must have that

$$\bigcup_{i=1,\dots,k-1} supp(\widehat{\sigma}_i) = \Delta_{mk} \tag{3.27}$$

We shall prove, by induction on $k \geq 4$, that

$$\left| \bigcup_{i=1,\dots,k-1} supp(\widehat{\sigma}_i) \right| = (k-1)supp(\omega) - (k-2)intersect(\omega) \tag{3.28}$$

If $k = 4$ then $\omega\colon B_4 =< \sigma_1, \sigma_2, \sigma_3 >\to S_{4m}$ and since, by assumption we have

$$supp(\widehat{\sigma}_1) \cap supp(\widehat{\sigma}_3) = \emptyset$$

and according to remark 3.19 we have

$$|supp(\widehat{\sigma}_1) \cap supp(\widehat{\sigma}_2)| = |supp(\widehat{\sigma}_2) \cap supp(\widehat{\sigma}_3)| = intersect(\omega)$$

all in all this means that

$$|supp(\widehat{\sigma}_1) \cup supp(\widehat{\sigma}_2) \cup supp(\widehat{\sigma}_3)| =$$

$$= |supp(\widehat{\sigma}_1)| + |supp(\widehat{\sigma}_2)| + |supp(\widehat{\sigma}_3)| - |supp(\widehat{\sigma}_1) \cap supp(\widehat{\sigma}_2)| -$$

$$-|supp(\widehat{\sigma}_2) \cap supp(\widehat{\sigma}_3)| = (k-1)supp(\omega) - (k-2)intersect(\omega)$$

and equation 3.28 holds in this case.

Now assume that equation 3.28 holds for $k-1$. Since by assumption we have that

$$supp(\widehat{\sigma}_{k-1}) \cap (\bigcup_{i=1,\ldots,k-3} supp(\sigma_i)) = \emptyset$$

and so

$$\left| supp(\widehat{\sigma}_{k-1}) \cap (\bigcup_{i=1,\ldots,k-2} supp(\sigma_i)) \right| = |supp(\widehat{\sigma}_{k-1}) \cap supp(\widehat{\sigma}_{k-2})| = intersect(\omega)$$

(see remark 3.19 for the last equality) hence we conclude that

$$\left| \bigcup_{i=1,\ldots,k-1} supp(\widehat{\sigma}_i) \right| = \left| supp(\widehat{\sigma}_{k-1}) \cup (\bigcup_{i=1,\ldots,k-2} supp(\sigma_i)) \right| =$$

$$= |supp(\widehat{\sigma}_{k-1})| + \left| \bigcup_{i=1,\ldots,k-2} supp(\sigma_i) \right| - \left| supp(\widehat{\sigma}_{k-1}) \cap (\bigcup_{i=1,\ldots,k-2} supp(\sigma_i)) \right| =$$

$$= |supp(\widehat{\sigma}_{k-1})| + \left| \bigcup_{i=1,\ldots,k-2} supp(\sigma_i) \right| - intersect(\omega) =$$

$$= |supp(\widehat{\sigma}_{k-1})| + ((k-2)supp(\omega) - (k-3)intersect(\omega)) - intersect(\omega) =$$

$$= (k-1)supp(\omega) - (k-2)intersect(\omega)$$

see remark 3.19 for the last equality.

Now since, by equation 3.27, $|\bigcup_{i=1,\ldots,k-1} supp(\widehat{\sigma}_i)| = mk$, then (3.28) implies that

$$(k-1)supp(\omega) - (k-2)intersect(\omega) = mk \tag{3.29}$$

let us rewrite this equation as follows

$$supp(\omega) = \frac{m(k-1) + m + (k-1)intersect(\omega) - intersect(\omega)}{k-1} =$$

$$= m + intersect(\omega) + \frac{m - intersect(\omega)}{k-1} \tag{3.30}$$

but since this equation involves only integers, there exists an $n \in \mathbb{Z}$ such that

$$\frac{m - intersect(\omega)}{k-1} = -n$$

or

$$intersect(\omega) = m + n(k-1) \tag{3.31}$$

Now by substituting the expression of $intersect(\omega)$ from equation 3.31 into equation 3.30 we get

$$supp(\omega) = m + (m + n(k-1)) - n = 2m + n(k-2)$$

and so we have the two equations

$$intersect(\omega) = m + n(k-1) \tag{3.32}$$
$$supp(\omega) = 2m + n(k-2) \tag{3.33}$$

Consider now $\widehat{\sigma}_1, \widehat{\sigma}_2$ and $\widehat{\sigma}_3$ (here we assume that $k \geq 4$). Since, by assumption we have that

$$supp(\widehat{\sigma}_1) \cap supp(\widehat{\sigma}_3) = \emptyset$$

which implies that

$$(supp(\widehat{\sigma}_1) \cap supp(\widehat{\sigma}_2)) \cap (supp(\widehat{\sigma}_2) \cap supp(\widehat{\sigma}_3)) = \emptyset$$

but since $|supp(\widehat{\sigma}_1) \cap supp(\widehat{\sigma}_2)| = |supp(\widehat{\sigma}_2) \cap supp(\widehat{\sigma}_3)| = intersect(\omega)$ and since $supp(\widehat{\sigma}_1) \cap supp(\widehat{\sigma}_2), supp(\widehat{\sigma}_2) \cap supp(\widehat{\sigma}_3) \subseteq supp(\widehat{\sigma}_2)$ we have that

$$2(intersect(\omega)) \leq supp(\omega),$$

but according to Corollary 3.21, we also have that

$$2(intersect(\omega)) \geq supp(\omega).$$

Hence

$$2(intersect(\omega)) = supp(\omega)$$

which, according to equations 3.32 and 3.33, translates to

$$2m + 2n(k-1) = 2m + n(k-2)$$

and this is equivalent to

$$nk = 0$$

which means that $n = 0$.

Hence we conclude from (3.32) and (3.33) that $intersect(\omega) = m$ and $supp(\omega) = 2m$.

□

3.4 Conjugacies between Model Homomorphisms

In this subsection we prove that there are non-trivial conjugacies between the model homomorphisms defined in Definition 3.9. Since we are interested in model homomorphisms up to conjugacy this will lead to a more concise definition of the model homomorphisms.

By definition 3.9, for each m there is only one model homomorphism $S_k \to B_{mk}$ which is good of type 1 (see definition 3.15). Hence, we will study the conjugacies between the model homomorphisms which are good of type 2, as defined in Definition 3.9.

We shall use the following notations (see notation 3.4 and notation 3.6) :

Fix a natural number $m \in \mathbb{N}$, $m \geq 2$.

For each $i \geq 1$ and $q \in \bigtriangledown_m$ we denote

$$a_q^i = 1 + m(i-1) + q \in supp(A_i^m).$$

Now let $\sigma \in \mathbf{S}(\{j, \dots, j+r-1\}) \cong S_r$ be some permutation and let $t_j, \dots, t_{j+r-1} \in \bigtriangledown_m$. We denote by $C^{\sigma,m}_{t_j,\dots,t_{j+r-1}}$ the permutation given by the formula

$$C^{\sigma,m}_{t_j,\dots,t_{j+r-1}}(a_p^i) = a^{\sigma(i)}_{|p+t_{\sigma(i)}|_m} \quad \text{for all } p \in \bigtriangledown_m \tag{3.34}$$

for each $i \in \{j, \dots, j+r-1\}$.

According to Lemma 3.5, we have that

$$(A_i^m)^{(C^{\sigma,m}_{t_j,\dots,t_{j+r-1}})^{-1}} = A^m_{\sigma(i)} \tag{3.35}$$

Lemma 3.23. *Let $m \in \mathbb{N}$, $m \geq 2$. Let $\sigma, \tau \in \mathbf{S}(\{j, \dots, j+r-1\}) \cong S_r$ be any two permutations and let $t^1_j, \dots, t^1_{j+r-1}, t^2_j, \dots, t^2_{j+r-1} \in \bigtriangledown_m$. Then the following formulas hold*

(a)

$$C^{\sigma,m}_{t^1_j,\dots,t^1_{j+r-1}} \cdot C^{\tau,m}_{t^2_j,\dots,t^2_{j+r-1}} = C^{\sigma\cdot\tau,m}_{t^3_j,\dots,t^3_{j+r-1}}$$

where

$$t_i^3 = |t^2_{\sigma^{-1}(i)} + t^1_i|_m \tag{3.36}$$

for each $i \in \{j, \ldots, j+r-1\}$.

(b)

$$(C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}})^{-1} = C^{\sigma^{-1},m}_{t^2_j,\ldots,t^2_{j+r-1}}$$

where

$$t^2_i = |-t^1_{\sigma(i)}|_m$$

for each $i \in \{j, \ldots, j+r-1\}$.

(c)

$$(C^{\tau,m}_{t^2_j,\ldots,t^2_{j+r-1}})^{-1} \cdot C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}} \cdot C^{\tau,m}_{t^2_j,\ldots,t^2_{j+r-1}} = C^{\tau^{-1}\cdot\sigma\cdot\tau,m}_{t^3_j,\ldots,t^3_{j+r-1}}$$

where

$$t^3_i = |t^2_{\sigma^{-1}(\tau(i))} + t^1_{\tau(i)} - t^2_{\tau(i)}|_m \tag{3.37}$$

for each $i \in \{j, \ldots, j+r-1\}$.

Proof.

In this proof we denote, for each $i \in \mathbb{N}$, $i \geq 1$

$$A_i = (1 + m(i-1), \ldots, mi)$$

(i.e., $A_i = A^m_i$ from notation 3.4 but we suppress the superscript).

Proof of (a) :

By (3.34) we have for each $i \in \{j, \ldots, j+r-1\}$ and $q \in \bigtriangledown_m$

$$C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}} \cdot C^{\tau,m}_{t^2_j,\ldots,t^2_{j+r-1}}(a^i_q) = C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}}(C^{\tau,m}_{t^2_j,\ldots,t^2_{j+r-1}}(a^i_q)) =$$

$$= C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}}(a^{\tau(i)}_{|q+t^2_{\tau(i)}|_m}) = a^{\sigma(\tau(i))}_{|q+t^2_{\tau(i)}+t^1_{\sigma(\tau(i))}|_m} = a^{\sigma\cdot\tau(i)}_{|q+t^2_{\tau(i)}+t^1_{\sigma\cdot\tau(i)}|_m}$$

which means, by (3.34), that

$$C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}} \cdot C^{\tau,m}_{t^2_j,\ldots,t^2_{j+r-1}} = C^{\sigma\cdot\tau,m}_{t^3_j,\ldots,t^3_{j+r-1}}$$

where

$$t^3_{\sigma\cdot\tau(i)} = |t^2_{\tau(i)} + t^1_{\sigma\cdot\tau(i)}|_m$$

or equivalently

$$t^3_i = |t^2_{\sigma^{-1}(i)} + t^1_i|_m$$

Proof of (b) :

For each $i \in \{j, \ldots, j+r-1\}$ set $t^2_i = |-t^1_{\sigma(i)}|_m$.

According to part (a) we have

$$C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}} \cdot C^{\sigma^{-1},m}_{t^2_j,\ldots,t^2_{j+r-1}} = C^{id,m}_{t^3_j,\ldots,t^3_{j+r-1}}$$

where

$$t^3_i = |t^2_{\sigma^{-1}(i)} + t^1_i|_m = ||-t^1_i|_m + t^1_i|_m = 0$$

Hence

$$C^{id,m}_{t^3_j,\ldots,t^3_{j+r-1}} = C^{id,m}_{0,\ldots,0} = id$$

and so

$$(C^{\sigma,m}_{t^1_j,\ldots,t^1_{j+r-1}})^{-1} = C^{\sigma^{-1},m}_{t^2_j,\ldots,t^2_{j+r-1}}$$

where $t^2_i = |-t^1_{\sigma(i)}|_m$.

Proof of (c) :

By part (b) we have that

$$(C^{\tau,m}_{t^2_j,\dots,t^2_{j+r-1}})^{-1} = C^{\tau^{-1},m}_{|-t^2_{\tau(j)}|_m,\dots,|-t^2_{\tau(j+r-1)}|_m}$$

and by part (a) and (3.36) we have that

$$C^{\sigma,m}_{t^1_j,\dots,t^1_{j+r-1}} \cdot C^{\tau,m}_{t^2_j,\dots,t^2_{j+r-1}} = C^{\sigma\cdot\tau,m}_{|t^2_{\sigma^{-1}(j)}+t^1_j|_m,\dots,|t^2_{\sigma^{-1}(j+r-1)}+t^1_{j+r-1}|_m}.$$

Using part (a) and (3.36) again we now have

$$(C^{\tau,m}_{t^2_j,\dots,t^2_{j+r-1}})^{-1} \cdot C^{\sigma,m}_{t^1_j,\dots,t^1_{j+r-1}} \cdot C^{\tau,m}_{t^2_j,\dots,t^2_{j+r-1}} =$$

$$= C^{\tau^{-1},m}_{|-t^2_{\tau(j)}|_m,\dots,|-t^2_{\tau(j+r-1)}|_m} \cdot C^{\sigma\cdot\tau,m}_{|t^2_{\sigma^{-1}(j)}+t^1_j|_m,\dots,|t^2_{\sigma^{-1}(j+r-1)}+t^1_{j+r-1}|_m} =$$

$$= C^{\tau^{-1}\cdot\sigma\cdot\tau,m}_{t^3_j,\dots,t^3_{j+r-1}}$$

where

$$t^3_i = |t^2_{\sigma^{-1}(\tau(i))} + t^1_{\tau(i)} - t^2_{\tau(i)}|_m$$

for each $i \in \{j, \dots, j+r-1\}$.

$\square$

Proposition 3.24. *Let $k, m \in \mathbb{N}$, $m \geq 2$. Let l be a divisor of m, $1 \leq l < m$, and for each $i = 1, \dots, k-1$ let $\{t^i_{(i-1)l+1}, \dots, t^i_{(i+1)l}\} \in \bigtriangledown_{\frac{m}{l}}$.*

Suppose that $\phi_{m,l,\{t^i_{(i-1)l+1},\dots,t^i_{(i+1)l}\}_{i=1,\dots,k-1}} \colon B_k \to S_{mk}$, defined by

$$\phi_{m,l,\{t^i_{(i-1)l+1},\dots,t^i_{(i+1)l}\}_{i=1,\dots,k-1}}(\sigma_i) = A_1^{\frac{m}{l}} \cdots A_{(i-1)l}^{\frac{m}{l}} \cdot C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\dots,t^i_{l(i+1)}} \cdot A_{(i+1)l+1}^{\frac{m}{l}} \cdots A_{kl}^{\frac{m}{l}} \quad (3.38)$$

for $i = 1, \dots, k-1$, is a homomorphism.

Then $\phi_{m,l,\{t^i_{(i-1)l+1},\dots,t^i_{(i+1)l}\}_{i=1,\dots,k-1}} \sim \phi_{m,l,\{t'^i_{(i-1)l+1},\dots,t'^i_{(i+1)l}\}_{i=1,\dots,k-1}}$ where for each $i = 1, \dots, k-1$ and q, $(i-1)l+1 \leq q \leq (i+1)l$

$$t'^i_q = \begin{cases} 0 & q \neq (i-1)l+1 \\ p & q = (i-1)l+1 \end{cases} \quad (3.39)$$

for some $p \in \bigtriangledown_{\frac{m}{l}}$.

Proof. For the purposes of the proof we denote $\phi_{m,l,\{t^i_{(i-1)l+1},\ldots,t^i_{(i+1)l}\}_{i=1,\ldots,k-1}}$ as ϕ, $\phi_{m,l,\{t'^i_{(i-1)l+1},\ldots,t'^i_{(i+1)l}\}_{i=1,\ldots,k-1}}$ as ϕ', and $\phi_{m,l,\{t^i_{(i-1)l+1},\ldots,t^i_{(i+1)l}\}_{i=1,\ldots,k-1}}(\sigma_i)$ as $\widehat{\sigma}_i$.

Furthermore, we use the notation 3.4, where for each $p \in \mathbb{N}$, $p \geq 2$, and each $i \geq 1$ we denote

$$A_i^p = (1 + p(i-1), \ldots, pi)$$

We shall denote the d-th power of A_i^p as $(A_i^p)^d$ for any $d \in \mathbb{Z}$.

Consider the following element in S_{mk} :

$$D = \prod_{1 \leq i \leq k} C^{A_i^l, \frac{m}{l}}_{s_{(i-1)l+1},\ldots,s_{il}}$$

where the s_i-s are numbers in $\bigtriangledown_{\frac{m}{l}}$, yet to be determined, and in each $C^{A_i, \frac{m}{l}}_{s_{(i-1)l+1},\ldots,s_{il}}$ we consider A_i as a permutation in $\mathbf{S}(\{(i-1)l+1,\ldots,il\}) \cong S_l$.

Let us now show that the conjugation of ϕ by D is a model homomorphism (see Definition 3.9).

We have that

$$(\widehat{\sigma}_i)^D = (C^{\psi_l(\sigma_i), \frac{m}{l}}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}})^D \cdot \prod_{j=1,\ldots,(i-1)l,(i+1)l+1,\ldots,kl} (A_j^{\frac{m}{l}})^D$$

First note that $C^{A_i^l, \frac{m}{l}}_{s_{(i-1)l+1},\ldots,s_{il}}$ permutes the cycles $A^{\frac{m}{l}}_{(i-1)l+1}, \ldots, A^{\frac{m}{l}}_{il}$ among themselves (see notation 3.6), i.e.,

$$(A^{\frac{m}{l}}_{(i-1)l+1} \cdots A^{\frac{m}{l}}_{il})^{C^{A_i^l, \frac{m}{l}}_{s_{(i-1)l+1},\ldots,s_{il}}} = A^{\frac{m}{l}}_{(i-1)l+1} \cdots A^{\frac{m}{l}}_{il}.$$

This means that for each $i = 1, \ldots, k-1$ we have

$$\prod_{j=1,\ldots,(i-1)l,(i+1)l+1,\ldots,kl} (A_j^{\frac{m}{l}})^D = \prod_{j=1,\ldots,(i-1)l,(i+1)l+1,\ldots,kl} A_j^{\frac{m}{l}} \tag{3.40}$$

Now note that since $(\psi_l(\sigma_i))^2 = A_i^l \cdot A_{i+1}^l$ for each $i = 1, \ldots, k-1$, $C^{A_j^l, \frac{m}{l}}_{s_{(j-1)l+1},\ldots,s_{jl}}$ is disjoint from $C^{\psi_l(\sigma_i), \frac{m}{l}}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}}$ except for $j = i, i+1$. Hence

$$(C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}})^D = (C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}})^{C^{A^l_i\cdot A^l_{i+1},\frac{m}{l}}_{s_{(i-1)l+1},\ldots,s_{(i+1)l}}} \tag{3.41}$$

(see notation 3.6 for the expression $C^{A^l_i\cdot A^l_{i+1},\frac{m}{l}}_{s_{(i-1)l+1},\ldots,s_{(i+1)l}}$).

Let us now compute the expression on the right hand side of (3.41).

According to part (c) of Lemma 3.23 and using the fact that $(\psi_l(\sigma_i))^2 = A^l_i \cdot A^l_{i+1}$ for each $i = 1, \ldots, k-1$, we have that

$$(C^{\psi_l(\sigma_i),\frac{m}{l}}_{t^i_{l(i-1)+1},\ldots,t^i_{l(i+1)}})^{C^{A^l_i\cdot A^l_{i+1},\frac{m}{l}}_{s_{(i-1)l+1},\ldots,s_{(i+1)l}}} = C^{\psi_l(\sigma_i),\frac{m}{l}}_{t'^i_{l(i-1)+1},\ldots,t'^i_{l(i+1)}} \tag{3.42}$$

where

$$t'^i_j = |s_{\psi_l(\sigma_i)(j)} + t^i_{\psi^2_l(\sigma_i)(j)} - s_{\psi^2_l(\sigma_i)(j)}|_{\frac{m}{l}} \tag{3.43}$$

for each $j = l(i-1)+1, \ldots, l(i+1)$.

The equations (3.40), (3.41) and (3.42) prove that ϕ^D is indeed a model homomorphism.

We now set the values of the $s_q \in \nabla_{\frac{m}{l}}$ so that the t'^i_q in (3.43) will satisfy the equations (3.39) in our claim.

We continue the proof in a series of steps as follows.

Step 1 : At this step we set the values of $s_1, \ldots, s_{2l}$.

First set $s_{\psi^2_l(\sigma_1)(1)} = 0$. Next set inductively the values of $s_{\psi^3_l(\sigma_1)(1)}, \ldots, s_{\psi^{2l}_l(\sigma_1)(1)}, s_{\psi_l(\sigma_1)(1)}$ by the following set of equations (recall that $\psi^{2l}_l(\sigma_i) = id$ for any i)

$$\begin{array}{rcl}
s_{\psi^3_l(\sigma_1)(1)} & = & |s_{\psi^2_l(\sigma_1)(1)} + t^1_{\psi^3_l(\sigma_1)(1)}|_{\frac{m}{l}} \\
 & \vdots & \\
s_{\psi^{j+2}_l(\sigma_1)(1)} & = & |s_{\psi^{j+1}_l(\sigma_1)(1)} + t^1_{\psi^{j+2}_l(\sigma_1)(1)}|_{\frac{m}{l}} \\
s_{\psi^{j+3}_l(\sigma_1)(1)} & = & |s_{\psi^{j+2}_l(\sigma_1)(1)} + t^1_{\psi^{j+3}_l(\sigma_1)(1)}|_{\frac{m}{l}} \\
 & \vdots & \\
s_{\psi^{2l}_l(\sigma_1)(1)} & = & |s_{\psi^{2l-1}_l(\sigma_1)(1)} + t^1_{\psi^{2l}_l(\sigma_1)(1)}|_{\frac{m}{l}} \\
s_{\psi_l(\sigma_1)(1)} & = & |s_{\psi^{2l}_l(\sigma_1)(1)} + t^1_{\psi_l(\sigma_1)(1)}|_{\frac{m}{l}}
\end{array} \tag{3.44}$$

where the j-th equation in (3.44) sets the value of $s_{\psi_l^j(\sigma_1)(1)}$ using the value of $s_{\psi_l^{j-1}(\sigma_1)(1)}$ defined in the $j-1$-th equation.

We now have, according to (3.43), that

$$t'^1_{\psi_l^j(\sigma_1)(1)} = |s_{\psi_l^{j+1}(\sigma_1)(1)} + t^1_{\psi_l^{j+2}(\sigma_1)(1)} - s_{\psi_l^{j+2}(\sigma_1)(1)}|_{\frac{m}{l}} =$$

$$= |s_{\psi_l^{j+1}(\sigma_1)(1)} + t^1_{\psi_l^{j+2}(\sigma_1)(1)} - (s_{\psi_l^{j+1}(\sigma_1)(1)} + t^1_{\psi_l^{j+2}(\sigma_1)(1)})|_{\frac{m}{l}} = 0$$

for each $j = 1, \ldots, 2l-1$ and furthermore

$$t'^1_1 = |s_{\psi_l(\sigma_1)(1)} + t^1_{\psi_l^2(\sigma_1)(1)} - s_{\psi_l^2(\sigma_1)(1)}|_{\frac{m}{l}} = |s_{\psi_l(\sigma_1)(1)} + t^1_{\psi_l^2(\sigma_1)(1)}|_{\frac{m}{l}}$$

Let us denote

$$p = |s_{\psi_l(\sigma_1)(1)} + t^1_{\psi_l^2(\sigma_1)(1)}|_{\frac{m}{l}} \in \nabla_{\frac{m}{l}}$$

Hence we now have that the t'^1_j satisfy (3.39) for each $j = 1, \ldots, 2l$. •

Step 2 : At this step we set the rest of the values : $s_{2l+1}, \ldots, s_{kl}$. We do this in $k-2$ steps where in step q we set the values of $s_{ql+1}, \ldots, s_{(q+1)l}$ where $q = 2, \ldots, k-1$.

For each $q = 2, \ldots, k-1$ we set

$$s_j = |s_{\psi_l^{-1}(\sigma_q)(j)} + t^q_j|_{\frac{m}{l}} \qquad \text{for} \quad j = ql+1, \ldots, (q+1)l$$

which is equivalent to

$$s_{\psi_l^2(\sigma_q)(j)} = |s_{\psi_l(\sigma_q)(j)} + t^q_{\psi_l^2(\sigma_q)(j)}|_{\frac{m}{l}} \qquad \text{for} \quad j = ql+1, \ldots, (q+1)l \tag{3.45}$$

Now according to (3.43) and (3.45), we have that

$$t'^q_j = |s_{\psi_l(\sigma_q)(j)} + t^q_{\psi_l^2(\sigma_q)(j)} - s_{\psi_l^2(\sigma_q)(j)}|_{\frac{m}{l}} =$$

$$= |s_{\psi_l(\sigma_q)(j)} + t^q_{\psi_l^2(\sigma_q)(j)} - (s_{\psi_l(\sigma_q)(j)} + t^q_{\psi_l^2(\sigma_q)(j)})|_{\frac{m}{l}} = 0$$

for each $q = 2, \ldots, k-1$ and $j = ql+1, \ldots, (q+1)l$, in accordance with (3.39). •

Step 3 : It is now left to show that the values of t'^q_j satisfy (3.39) for $q = 2, \ldots, k-1$ and $j = (q-1)l+1, \ldots, ql$.

Since $\phi^D = \phi'$ is a model homomorphism, as we have shown above, we have by Lemma 3.12 and Lemma 3.13 that

$$\forall j \in \triangledown_l \quad t'^i_{(i-1)l+1+|1+j|_l} + t'^i_{il+1+|1+j|_l} \equiv t'^{i+1}_{(i+1)l+1+j} + t'^{i+1}_{il+1+|1+j|_l} (\mathrm{mod}\ \frac{m}{l}) \tag{3.46}$$

for each $i = 1, \ldots, k-2$. But since, by step 1 and step 2, we have that $t'^i_{il+1+|1+j|_l} = 0$ for each $i = 1, \ldots, k-1$ and $j \in \triangledown_l$, and since each summand in (3.46) is in $\triangledown_{\frac{m}{l}}$, we have that (3.46) implies

$$\forall j \in \triangledown_l \quad t'^i_{(i-1)l+1+|1+j|_l} = t'^{i+1}_{il+1+|1+j|_l} \tag{3.47}$$

Now since, by step 1, $t'^1_1 = p$ and $t'^1_2 = \cdots = t'^1_l = 0$, (3.47) implies that for each $i = 2, \ldots, k-1$ we have

$$t'^i_{(i-1)l+1} = p \qquad \text{and} \qquad t'^i_{(i-1)l+2} = \cdots = t'^i_{il} = 0$$

•

Summing up, we have shown that $\phi' = \phi^D$ is a model homomorphism and in steps 1, 2 and 3 we have shown that the values of t'^i_q in ϕ' satisfy (3.39) which completes the proof.

□

Proposition 3.25. *Let $k, m \in \mathbb{N}$, $m \geq 2$. Let l be a divisor of m, $1 \leq l < m$.*

For each $p \in \triangledown_{\frac{m}{l}}$ let l_p be the minimal number between 1 and $\frac{m}{l}$ such that

$$l_p \cdot p \equiv 0 \ (mod\ \frac{m}{l}). \tag{3.48}$$

Suppose that

$$2l \cdot l_p \neq \frac{m}{l} \qquad \textit{for all} \quad p \in \triangledown_{\frac{m}{l}} \tag{3.49}$$

((3.49) holds for example when $\frac{m}{l}$ is odd). Then there are exactly $\frac{m}{l}+1$ pairwise non-conjugate model homomorphisms $B_k \to S_{mk}$ in the set of model homomorphisms defined in definition 3.9.

Explicitly, these homomorphisms are

$$\psi_m(\sigma_i) = C_{1,0}^{(i,i+1),m}$$

and

$$\phi_{m,l,\{p,0,\ldots,0\}_{i=1,\ldots,k-1}}(\sigma_i) = A_1^{\frac{m}{l}} \cdots A_{(i-1)l}^{\frac{m}{l}} \cdot C_{p,0,\ldots,0}^{\psi_l(\sigma_i),\frac{m}{l}} \cdot A_{(i+1)l+1}^{\frac{m}{l}} \cdots A_{kl}^{\frac{m}{l}} \tag{3.50}$$

for each $p = 0, \ldots, \frac{m}{l} - 1$.

Proof. According to proposition 3.24, every model homomorphism which is good of type 2 (see definition 3.15) is conjugate to a homomorphism of the form (3.50) for some $p \in \bigtriangledown_{\frac{m}{l}}$.

Let ϕ_1 and ϕ_2 be two homomorphisms of the form (3.50) with p_1 and p_2 instead of p respectively. Suppose that there is an element $D \in S_{mk}$ such that $\phi_1^D = \phi_2$. We will prove that necessarily $p_1 = p_2$ which will prove our claim.

Let us denote $\phi_1(\sigma_i)$ as $\widehat{\sigma}_i$.

Let us write the equations $(\widehat{\sigma}_i)^D = \phi_2(\sigma_i)$ for $i = 1, 3$ as follows :

$$(C_{p_1,0,\ldots,0}^{\psi_l(\sigma_1),\frac{m}{l}})^D \cdot (\prod_{j=2l+1,\ldots,kl} (A_j^{\frac{m}{l}}))^D = C_{p_2,0,\ldots,0}^{\psi_l(\sigma_1),\frac{m}{l}} \cdot \prod_{j=2l+1,\ldots,kl} (A_j^{\frac{m}{l}}) \tag{3.51}$$

$$(C_{p_1,0,\ldots,0}^{\psi_l(\sigma_3),\frac{m}{l}})^D \cdot (\prod_{j=1,\ldots,2l,4l+1,\ldots,kl} (A_j^{\frac{m}{l}}))^D = C_{p_2,0,\ldots,0}^{\psi_l(\sigma_3),\frac{m}{l}} \cdot \prod_{j=1,\ldots,2l,4l+1,\ldots,kl} (A_j^{\frac{m}{l}}) \tag{3.52}$$

According to lemma 3.5, the cyclic decomposition of each $C_{p_1,0,\ldots,0}^{\psi_l(\sigma_i),\frac{m}{l}}$ consists of exactly $\frac{m}{l} \cdot \frac{1}{l_{p_1}}$ $(2l \cdot l_{p_1})$-cycles where l_p is defined in (3.48). Since, according to our assumption (3.49), $2l \cdot l_{p_1} \neq \frac{m}{l}$ we have that each side of the equations (3.51) and (3.52) consists of a product between disjoint permutations whose cyclic decompositions contain cycles of different lengths.

In particular, (3.51) implies

$$(\prod_{j=2l+1,\ldots,kl} (A_j^{\frac{m}{l}}))^D = \prod_{j=2l+1,\ldots,kl} (A_j^{\frac{m}{l}}) \tag{3.53}$$

and (3.52) implies

$$(\prod_{j=1,\ldots,2l,4l+1,\ldots,kl} (A_j^{\frac{m}{l}}))^D = \prod_{j=1,\ldots,2l,4l+1,\ldots,kl} (A_j^{\frac{m}{l}}). \tag{3.54}$$

Now (3.53) implies in particular that

$$\{(A_{4l+1}^{\frac{m}{l}})^D, \dots, (A_{kl}^{\frac{m}{l}})^D\} \subseteq \{A_{2l+1}^{\frac{m}{l}}, \dots, A_{kl}^{\frac{m}{l}}\}$$

and (3.54) implies

$$\{(A_{4l+1}^{\frac{m}{l}})^D, \dots, (A_{kl}^{\frac{m}{l}})^D\} \subseteq \{A_1^{\frac{m}{l}}, \dots, A_{2l}^{\frac{m}{l}}\} \cup \{A_{4l+1}^{\frac{m}{l}}, \dots, A_{kl}^{\frac{m}{l}}\}$$

Hence

$$\{(A_{4l+1}^{\frac{m}{l}})^D, \dots, (A_{kl}^{\frac{m}{l}})^D\} \subseteq \{A_{2l+1}^{\frac{m}{l}}, \dots, A_{kl}^{\frac{m}{l}}\} \cap (\{A_1^{\frac{m}{l}}, \dots, A_{2l}^{\frac{m}{l}}\} \cup \{A_{4l+1}^{\frac{m}{l}}, \dots, A_{kl}^{\frac{m}{l}}\})$$

or equivalently

$$\{(A_{4l+1}^{\frac{m}{l}})^D, \dots, (A_{kl}^{\frac{m}{l}})^D\} \subseteq \{A_{4l+1}^{\frac{m}{l}}, \dots, A_{kl}^{\frac{m}{l}}\}$$

but since the A_i-s are disjoint this means that

$$\{(A_{4l+1}^{\frac{m}{l}})^D, \dots, (A_{kl}^{\frac{m}{l}})^D\} = \{A_{4l+1}^{\frac{m}{l}}, \dots, A_{kl}^{\frac{m}{l}}\} \tag{3.55}$$

hence (3.54) and (3.55) imply that

$$(\prod_{j=1,\dots,2l} (A_j^{\frac{m}{l}}))^D = \prod_{j=1,\dots,2l} (A_j^{\frac{m}{l}}).$$

Hence, by lemma 3.5 we can write

$$D|_{supp(\bigcup_{j=1,\dots,2l} A_j)} = C_{s_1,\dots,s_{2l}}^{\tau,\frac{m}{l}} \tag{3.56}$$

where $\tau \in S_{2l}$.

Furthermore, by (3.51), we have that

$$(C_{p_1,0,\dots,0}^{\psi_l(\sigma_1),\frac{m}{l}})^D = C_{p_2,0,\dots,0}^{\psi_l(\sigma_1),\frac{m}{l}}$$

which, by (3.56), can be rewritten as

$$(C_{p_1,0,\dots,0}^{\psi_l(\sigma_1),\frac{m}{l}})^{C_{s_1,\dots,s_{2l}}^{\tau,\frac{m}{l}}} = C_{p_2,0,\dots,0}^{\psi_l(\sigma_1),\frac{m}{l}} \tag{3.57}$$

According to part (c) of lemma 3.23, (3.57) implies that

$$(\psi_l(\sigma_1))^\tau = \psi_l(\sigma_1)$$

but since $\psi_l(\sigma_1)$ is a $2l$-cycle, then according to part (b) of lemma 5.3 we have that

$$\tau = \psi_l^q(\sigma_1) \qquad \text{for some } q,\ 0 \le q \le 2l-1.$$

Hence we can rewrite (3.57) as

$$(C^{\psi_l(\sigma_1),\frac{m}{l}}_{p_1,0,\dots,0})^{C^{\psi_l^q(\sigma_1),\frac{m}{l}}_{s_1,\dots,s_{2l}}} = C^{\psi_l(\sigma_1),\frac{m}{l}}_{p_2,0,\dots,0} \tag{3.58}$$

In order to prove our claim it is enough to show that equation (3.58) has a solution $s_1,\dots,s_{2l} \in \bigtriangledown_{\frac{m}{l}}$ only if $p_1 = p_2$.

Suppose then that (3.58) has a solution $s_1,\dots,s_{2l} \in \bigtriangledown_{\frac{m}{l}}$.

Let us rewrite (3.37) from part (c) of lemma 3.23 as

$$t^3_{\tau^{-1}(i)} = |t^2_{\sigma^{-1}(i)} + t^1_i - t^2_i|_m \tag{3.59}$$

(the m in (3.59) is $\frac{m}{l}$ in our case, τ is $\psi_l^q(\sigma_1)$ and σ is $\psi_l(\sigma_1)$).

Then (3.58) and (3.59) imply the following set of $2l$ equations :

$$\begin{array}{rcl} p_2 & = & |s_{\psi_l^{-1}(\sigma_1)(1)} + 0 - s_1|_{\frac{m}{l}} \\ 0 & = & |s_{\psi_l^{-1}(\sigma_1)(2)} + 0 - s_2|_{\frac{m}{l}} \\ & \vdots & \\ 0 & = & |s_{\psi_l^{-1}(\sigma_1)(\psi_l^{-q}(\sigma_1)(1))} + p_1 - s_2|_{\frac{m}{l}} \\ & \vdots & \\ 0 & = & |s_{\psi_l^{-1}(\sigma_1)(2l)} + 0 - s_{2l}|_{\frac{m}{l}} \end{array} \tag{3.60}$$

Now the set of $2l$ equations (3.60) can be rewritten in matrix form as follows

$$(P-I)\begin{pmatrix} s_1 \\ \vdots \\ s_{2l} \end{pmatrix} \equiv p_2 \cdot \vec{e}_1 - p_1 \cdot \vec{e}_{\psi_l^{-q}(\sigma_1)(1)}\ (mod\ \frac{m}{l}) \tag{3.61}$$

where I is the unit matrix of size $2l$, $\vec{e}_j$ is the unit vector of size $2l$ (it has 1 in the j-th place and 0-s elsewhere) and P is the permutation matrix of size $2l$ corresponding to $\psi_l^{-1}(\sigma_1)$, i.e., its j-th row consists of zeros except for the $\psi_l^{-1}(\sigma_1)(j)$ column where the entry is 1.

Now note that since $\psi_l(\sigma_1)$ is a $2l$-cycle, its only power that has a fixed point is the identity, and so $\psi_l^{-1}(\sigma_1)(i) \neq i$ for all $i = 1, \ldots, 2l$. Hence the j-th column of the matrix $P - I$ in (3.61) has the entry -1 on the j-th row, it has the entry 1 on the $\psi_l^{-1}(\sigma_1)(j)$-th row (which is $\neq j$) and 0 in the rest of the entries. This means that the submodule of $(\mathbb{Z}_{\frac{m}{l}})^{2l}$ spanned by the columns of the matrix $P - I$ consists of vectors whose entries sum is 0. Hence (3.60) can have a solution only if $p_1 = p_2$.

□

4 Homomorphisms $\omega \colon B_k \to S_{mk}$ with $supp(\omega) = 2m$

Recall the definition of model homomorphisms $\omega \colon B_k \to S_{mk}$ with $supp(\omega) = 2m$:

$$\psi_m(\sigma_i) = C_{1,0}^{(i,i+1),m}$$

(see Definition 3.9 and Notations 3.6 and 3.4). Alternatively, we can write

$$\psi_m(\sigma_i) = (a_0^i, a_0^{i+1}, a_1^i, a_1^{i+1}, \ldots, a_{m-1}^i, a_{m-1}^{i+1})$$

where

$$a_j^i = 1 + m(i-1) + j \in supp(A_i^m) \quad j \in \bigtriangledown_m.$$

Recall (see Definition 3.9) that a homomorphism $B_k \to S_n$ is called standard if it is conjugate to a model homomorphism.

Theorem 4.1. *Let $k, m \in \mathbb{N}$ be such that $k > 6$ and $m \geq 2$. Every good transitive non-cyclic homomorphism $\omega \colon B_k \to S_{mk}$ with $supp(\omega) = 2m$ is standard.*

(The condition $supp(\omega) = 2m$ corresponds to case 1 of Definition 3.15 according to Lemma 3.22)

Proof. Let $\omega \colon B_k \to S_{mk}$ be a good transitive non-cyclic homomorphism with $supp(\omega) = 2m$.

We will define a permutation $\theta \in S_{mk}$ such that $\omega^\theta = \psi_m$ (where ψ_m is a model homomorphism defined in Definition 3.9). Recall that ω^θ is defined by conjugating each element in the image of ω by θ, i.e.

$$\omega^{\theta}(\sigma_i) = (\omega(\sigma_i))^{\theta} = (\widehat{\sigma}_i)^{\theta} \quad i = 1, \ldots, k-1$$

We shall prove our claim in five steps as follows :

Step 1: *intersect*$(\varphi) = m$.

Proof of Step 1: This follows directly from Lemma 3.22 since $supp(\omega) = 2m$ and ω is good.

Step 2 : For each $i = 1, \ldots, k-1$, each cycle D in the cycle decomposition of $\widehat{\sigma}_i$ is of even length and $|supp(D) \cap supp(\widehat{\sigma}_{i\pm 1})| = \frac{1}{2}|supp(D)|$ (as in Lemma 3.20, for $i = 1$ and $i = k-1$ we take $\widehat{\sigma}_{1\pm 1}$ and $\widehat{\sigma}_{(k-1)\pm 1}$ to mean $\widehat{\sigma}_2$ and $\widehat{\sigma}_{k-2}$ respectively).

Proof of Step 2 : Let $i \in \{1, \ldots, k-1\}$ and let us write

$$\widehat{\sigma}_i = D_1 \cdots D_r$$

according to Corollary 3.21, for each $j = 1, \ldots, r$ we have that $|supp(D_j) \cap supp(\widehat{\sigma}_{i\pm 1})| \geq \frac{1}{2} supp(D_j)$ and so

$$intersect(\omega) = |supp(\widehat{\sigma}_i) \cap supp(\widehat{\sigma}_{i\pm 1})| =$$

$$= \sum_{j=1,\ldots,r} |supp(D_j) \cap supp(\widehat{\sigma}_{i\pm 1})| \geq \sum_{j=1,\ldots,r} \frac{1}{2}|supp(D_j)| =$$

$$= \frac{1}{2}|supp(\widehat{\sigma}_i)| = m$$

but since $intersect(\omega) = m$ by step 1, we get that for each $j = 1, \ldots, r$: $|supp(D_j) \cap supp(\widehat{\sigma}_{i\pm 1})| = \frac{1}{2}|supp(D_j)|$ and $|supp(D_j)|$ must be even, i.e., each D_j is of even length.

Step 3: Let $i \in \{1, \ldots, k-2\}$ and let D be a cycle in the cycle decomposition of $\widehat{\sigma}_i$ and let us write D as follows (recall that D is of even length by step 2)

$$D = (x_0^i, x_0^{i+1}, x_1^i, x_1^{i+1}, \ldots, x_r^i, x_r^{i+1})$$

assume w.l.o.g that $x_0^i \notin supp(\widehat{\sigma}_{i+1})$ (this can be assumed by Step 2 since $|supp(D) \cap supp(\widehat{\sigma}_{i+1})| = \frac{1}{2}|supp(D)|$), then

$$\{x_0^i, x_1^i, \ldots, x_r^i\} \cap supp(\widehat{\sigma}_{i+1}) = \emptyset$$

and $D^{\widehat{\sigma}_{i+1}^{-1}\widehat{\sigma}_i^{-1}}$ can be written as follows

$$D^{\widehat{\sigma}_{i+1}^{-1}\widehat{\sigma}_i^{-1}} = (x_0^{i+1}, x_0^{i+2}, x_1^{i+1}, x_1^{i+2}, \dots, x_r^{i+1}, x_r^{i+2})$$

where $\{x_0^{i+2}, x_1^{i+2}, \dots, x_r^{i+2}\} \cap supp(\widehat{\sigma}_i) = \emptyset$

Proof of Step 3: Let us first consider D

$$D = (x_0^i, x_0^{i+1}, x_1^i, x_1^{i+1}, \dots, x_r^i, x_r^{i+1})$$

Consider $(x_0^i, x_0^{i+1}, x_1^i, x_1^{i+1}, \dots, x_r^i, x_r^{i+1})$ as an ordered tuple of (distinct) elements in Δ_{mk}. According to Lemma 3.20 at least one element in each consecutive pair in this tuple belongs to $supp(\widehat{\sigma}_{i+1})$ and according to Step 2 exactly half of the elements of this tuple belong to $supp(\widehat{\sigma}_{i+1})$. This means that in each consecutive pair in this tuple exactly one element belongs to $supp(\widehat{\sigma}_{i+1})$. Since, according to our assumption, $x_0^i \notin supp(\widehat{\sigma}_{i+1})$, we conclude that

$$\{x_0^i, x_1^i, \dots, x_r^i\} \cap supp(\widehat{\sigma}_{i+1}) = \emptyset$$

and $x_j^{i+1} \in supp(\widehat{\sigma}_{i+1})$ for each $j = 0, \dots, r$.

This means that $\widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_j^i) = \widehat{\sigma}_i(x_j^i) = x_j^{i+1}$ for each $j = 0, \dots, r$.

Let us now compute $D^{\widehat{\sigma}_{i+1}^{-1}\widehat{\sigma}_i^{-1}}$ directly

$$D^{\widehat{\sigma}_{i+1}^{-1}\widehat{\sigma}_i^{-1}} = (\widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_0^i), \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_0^{i+1}), \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_1^i), \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_1^{i+1}), \dots, \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_r^i), \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_r^{i+1})) =$$

$$= (x_0^{i+1}, \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_0^{i+1}), x_1^{i+1}, \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_1^{i+1}), \dots, x_r^{i+1}, \widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_r^{i+1}))$$

Let us denote $\widehat{\sigma}_i\widehat{\sigma}_{i+1}(x_j^{i+1})$ as x_j^{i+2} for $j = 0, \dots, r$. According to step 2 we have that $D^{\widehat{\sigma}_{i+1}\widehat{\sigma}_i}$, which is a cycle of $\widehat{\sigma}_{i+1}$, satisfies $|supp(D^{\widehat{\sigma}_{i+1}\widehat{\sigma}_i}) \cap supp(\widehat{\sigma}_i)| = \frac{1}{2}|supp(D^{\widehat{\sigma}_{i+1}\widehat{\sigma}_i})|$ and so, since $x_j^{i+1} \in supp(\widehat{\sigma}_i)$ for each $j = 0, \dots, r$,this means that

$$\{x_0^{i+2}, x_1^{i+2}, \dots, x_r^{i+2}\} \cap supp(\widehat{\sigma}_i) = \emptyset$$

Step 4: The cyclic type of $\widehat{\sigma}_1$ (and hence of $\widehat{\sigma}_i$ for all $i = 1, \dots, k-1$) is $[2m]$ (i.e. the cycle decomposition of each $\widehat{\sigma}_i$ consists of a single $2m$-cycle)

Proof of Step 4: Let D_1 be a cycle in the cycle decomposition of $\widehat{\sigma}_1$. Let us write D_1 as follows (recall that D_1 is of even length by step 2)

$$D_1 = (x_0^1, x_0^2, x_1^1, x_1^2, \ldots, x_r^1, x_r^2)$$

where we assume w.l.o.g that $x_0^1 \notin supp(\widehat{\sigma}_2)$ (this can be assumed by Step 2 since $|supp(D) \cap supp(\widehat{\sigma}_{i+1})| = \frac{1}{2}|supp(D)|$ for each $i = 1, \ldots, k-2$)

For $i = 1, \ldots, k-2$ define inductively

$$D_{i+1} = D_i^{\widehat{\sigma}_{i+1}^{-1}\widehat{\sigma}_i^{-1}}$$

since $\widehat{\sigma}_i^{\widehat{\sigma}_{i+1}^{-1}\widehat{\sigma}_i^{-1}} = \widehat{\sigma}_{i+1}$ we have that each D_i is a cycle in the cycle decomposition of $\widehat{\sigma}_i$ and according to step 3 we can write

$$D_i = (x_0^i, x_0^{i+1}, x_1^i, x_1^{i+1}, \ldots, x_r^i, x_r^{i+1})$$

where $x_j^i \notin supp(\widehat{\sigma}_{i+1})$ and $x_j^{i+1} \in supp(\widehat{\sigma}_{i+1})$ for $j = 0, \ldots, r$.

Now let us prove that the orbit of $x_0^1 \in \Delta_{mk}$ inside the permutation group generated by $\{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{k-1}\}$, denoted $orbit(x_0^1)$, is equal to the union of the supports of the D_i, i.e.,

$$orbit(x_0^1) = \bigcup_{i=1,\ldots,k-1} supp(D_i).$$

Let us denote the orbit of $x_0^1 \in \Delta_{mk}$ inside the permutation group generated by $\{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{j-1}\}$ as $orbit_j(x_0^1)$, so $orbit(x_0^1) = orbit_{k-1}(x_0^1)$. The proof will be carried out by induction on j.

The induction base is clear since the orbit of x_0^1 inside the group generated by $\widehat{\sigma}_1$ is exactly $supp(D_1)$. Now assume the induction hypothesis for $j-1$ and prove for j. Since ω is good (case 1 of Definition 3.15), we have that

$$supp(\widehat{\sigma}_j) \cap \bigcup_{i=1,\ldots,j-2} supp(\widehat{\sigma}_i) = \emptyset \tag{4.1}$$

and in particular

$$supp(D_j) \cap \bigcup_{i=1,\ldots,j-2} supp(\widehat{\sigma}_i) = \emptyset. \tag{4.2}$$

Furthermore, by step 3 we have that

$$supp(D_j) \cap supp(\widehat{\sigma}_{j-1}) = \{x_0^j, \ldots, x_r^j\} \tag{4.3}$$

and so, it follows from (4.2) and (4.3) that

$$supp(D_j) \cap \bigcup_{i=1,\ldots,j-1} supp(\widehat{\sigma}_i) =$$

$$= (supp(D_j) \cap \bigcup_{i=1,\ldots,j-2} supp(\widehat{\sigma}_i)) \cup (supp(D_j \cap supp(\widehat{\sigma}_{j-1})) =$$

$$= \{x_0^j, \ldots, x_r^j\} \tag{4.4}$$

and from (4.1) it follows that for any other cycle E_j, $E_j \neq D_j$, in the cycle decomposition of $\widehat{\sigma}_j$

$$supp(E_j) \cap \bigcup_{i=1,\ldots,j-1} supp(D_i) = \emptyset. \tag{4.5}$$

Now according to the induction hypothesis we have that $orbit_{j-1}(x_0^1) = \bigcup_{i=1,\ldots,j-1} supp(D_i)$ and by (4.4) and (4.5) we have that $orbit_{j-1}(x_0^1)$ has a non-trivial intersection only with the cycle D_j in $\widehat{\sigma}_j$. This means that

$$orbit_j(x_0^1) = \bigcup_{i=1,\ldots,j-1} supp(D_i) \cup supp(D_j) = \bigcup_{i=1,\ldots,j} supp(D_i)$$

which concludes the induction proof. We now have that

$$|orbit(x_0^1)| = |\bigcup_{i=1,\ldots,k-1} supp(D_i)| = |supp(D_1)| + \sum_{j=2,\ldots,k-1} |supp(D_j) - supp(D_{j-1})| =$$

$$= (2r+2) + (k-2)(r+1)$$

but since ω is transitive we must have that

$$2r + 2 + (k-2)(r+1) = mk$$

which is equivalent to

$$k(r+1) = mk$$

and we conclude that $r = m - 1$ and each $\widehat{\sigma}_i$ is a single $2m$-cycle.

Step 5: There exists a permutation $\theta \in S_{mk}$ such that $\omega^\theta = \psi_m$ (ψ_m is the model homomorphism defined in Definition 3.9).

Proof of Step 5: By step 4, each $\widehat{\sigma}_i$ for $i = 1, \ldots, k-1$ is a single $2m$-cycle. Let us write $\widehat{\sigma}_1$ explicitly

$$\widehat{\sigma}_1 = (x_0^1, x_0^2, x_1^1, x_1^2, \ldots, x_{m-1}^1, x_{m-1}^2)$$

where we assume that $x_0^1 \notin supp(\widehat{\sigma}_2)$ (this can be assumed by Step 2 since $|supp(D) \cap supp(\widehat{\sigma}_{i+1})| = \frac{1}{2}|supp(D)|$ for each $i = 1, \ldots, k-2$).

Since $\widehat{\sigma}_i^{\widehat{\sigma}_{i+1}^{-1}\widehat{\sigma}_i^{-1}} = \widehat{\sigma}_{i+1}$ for each $i = 1, \ldots, k-2$, step 3 implies that we can write

$$\widehat{\sigma}_i = (x_0^i, x_0^{i+1}, x_1^i, x_1^{i+1}, \ldots, x_{m-1}^i, x_{m-1}^{i+1})$$

where $x_j^i \notin supp(\widehat{\sigma}_{i+1})$ and $x_j^{i+1} \in supp(\widehat{\sigma}_{i+1})$ for $j = 0, \ldots, m-1$. This means that $\{x_0^i, \ldots, x_{m-1}^i\}_{i=1,\ldots,k}$ is a set of mk distinct elements, i.e. it is equal to Δ_{mk}. Hence we can define the permutation $\theta \in S_{mk}$ as follows

$$\theta(a_j^i) = x_j^i \quad j = 0, \ldots, m-1 \quad i = 1, \ldots, k$$

where

$$a_j^i = 1 + m(i-1) + j \in supp(A_i^m) \quad j \in \bigtriangledown_m$$

which would give $\omega^\theta = \psi_m$ (see Definition 3.9).

□

5 Good Transitive Homomorphisms $\omega\colon B_k \to S_{3k}$

In this section we prove that good transitive homomorphisms $\omega\colon B_k \to S_{mk}$ are standard for $m = 3$, i.e. they are conjugate to the model homomorphisms (see Definition 3.9).

Notation 5.1. *Recall notation 3.4 in the case $m = 3$:*

$$A_i^3 = (3i-2, 3i-1, 3i) \qquad i \geq 1 \tag{5.1}$$

In this section we shall denote A_i^3 simply as A_i and we denote

$$(A_i \cdot A_i =) A_i^{-1} = (3i-2, 3i, 3i-1) \qquad i \geq 1 \tag{5.2}$$

(i.e. we use the superscripts of the A_i-s in the conventional manner instead of their use in notation 3.4). $\bullet$

We shall use the following important results from [Lin04b]

Lemma 5.2 (cf. Lemma 7.12 in [Lin04b]). *Let $k > 6$ and let $\omega \colon B_k \to \mathbf{S}_n$ be a homomorphism such that all components of $\widehat{\sigma}_1$ (including the degenerate component $Fix(\widehat{\sigma}_1)$) are of lengths at most $k-3$. Then ω is cyclic.*

Lemma 5.3 (cf. [Ar47b] and Lemma 2.7 in [Lin04b]). *Suppose that $A, B \in \mathbf{S}_n$ and $AB = BA$. Then:*

a) The set $supp(A)$ is B-invariant;

b) If for some r, $2 \leq r \leq n$, the r-component of A consists of a single r-cycle C, then the set $supp(C)$ is B-invariant and $B|supp(C) = C^q$ for some integer q, $0 \leq q < r$.

Lemma 5.4 (Normalization assumption). *Let $\omega \colon B_k \to S_n$ be a homomorphism and assume $\mathfrak{D}_r = \{A_j^r, \ldots, A_{j+t-1}^r\}$ (for some $j \geq 1$, see notation 3.4) be an r-subcomponent of $\widehat{\sigma}_1$ (of $\widehat{\sigma}_{k-1}$). Let $\Omega_{\mathfrak{D}_r}$ be the (co)retraction of ω to $\mathfrak{D}_r$ (assuming it is well defined). Suppose $\Omega_{\mathfrak{D}_r}$ is conjugate to Ω', then ω is conjugate to $\omega^C = \omega'$ for some $C \in \mathbf{S}(supp(\mathfrak{D}_r))$ where the (co)retraction of ω' to $\mathfrak{D}_r$ is equal to Ω' and the conjugation of the cycles of $\mathfrak{D}_r$ by C induces a permutation between them.*

Proof. We will prove the claim only for retractions, the dual claim (for coretractions) being completely similar.

Assume then that $\mathfrak{D}_r = \{A_j^r, \ldots, A_{j+t-1}^r\}$ is an r-subcomponent of $\widehat{\sigma}_1$ and denote $\Omega_{\mathfrak{D}_r} = \Omega$. Recall that by definition $\Omega(\sigma_l) \in \mathbf{S}(\{j, \ldots, j+t-1\}) \simeq S_t$ for each $l = 3, \ldots, k-1$. Suppose $\Omega' = \eta\Omega\eta^{-1}$ for some $\eta \in \mathbf{S}(\{j, \ldots, j+t-1\}) \simeq S_t$.

By using Lemma 3.5, we construct a permutation $C \in \mathbf{S}(\bigcup_{s=j,\ldots,j+t-1} supp(A_s))$ that corresponds to η and satisfies

$$(A_p^r)^C = A_{\eta(p)}^r \quad p \in supp(\eta)$$

And so, conjugation of the cycles $\mathfrak{D}_r = \{A_j^r, \ldots, A_{j+t-1}^r\}$ by C induces a permutation between them. Furthermore, we have for each $l = 3, \ldots, k-1$

$$(A_p^r)^{C^{-1}\widehat{\sigma}_l C} = (A_{\eta\Omega(\sigma_l)\eta^{-1}(p)}^r) \quad \text{for } j \leq p \leq j+t-1$$

or equivalently

$$(A_p^r)^{(\widehat{\sigma}_l)^C} = (A^r_{\Omega'(\sigma_l)(p)}) \quad \text{for } j \le p \le j+t-1$$

which means that the retraction of ω^C to $\mathfrak{D}_r$ is Ω'.

$\square$

Lemma 5.5. *Let $\omega\colon B_k \to S_k$ be a non-cyclic homomorphism such that $\omega(\sigma_i) = (i, i+1)$ for $i \neq k-2$, then either $\omega(\sigma_{k-2}) = (k-2, k-1)$ or $\omega(\sigma_{k-2}) = (k-2, k)$*

Proof. If $\omega(\sigma_{k-2})$ were disjoint from $\omega(\sigma_{k-3})$ or $\omega(\sigma_{k-1})$ it would commute with either of them and then ω would be cyclic according to Lemma 2.10 and Fact 2.9, contradicting our assumption. For the same reason, $\omega(\sigma_{k-2})$ could not be equal to either $\omega(\sigma_{k-3})$ or $\omega(\sigma_{k-1})$. This leaves only 4 possibilities for $\omega(\sigma_{k-2})$: $(k-3, k-1)$, $(k-3, k)$, $(k-2, k-1)$ or $(k-2, k)$. $(k-3, k-1)$ and $(k-3, k)$ are impossible since they do not commute with $\omega(\sigma_{k-4}) = (k-4, k-3)$. $(k-2, k-1)$ and $(k-2, k)$ are readily seen to be appropriate possibilities. $\square$

We are now ready to prove

Theorem 5.6. *For $k > 8$ every good transitive non-cyclic homomorphism $\omega\colon B_k \to S_{3k}$ is standard.*

Proof. Since ω is good by assumption, then according to Lemma 3.22 there are only two possibilities : Either $supp(\omega) = 6$ or $supp(\omega) = 3k$. It was proved in Theorem 4.1 (for $m = 3$) that ω is standard in the case $supp(\omega) = 6$. Assume then that $supp(\omega) = 3k$. Then the degenerate component of $\widehat{\sigma}_1$ is empty. Furthermore, by Lemma 3.22 we have in this case $intersect(\omega) = 3k$.

In the rest of the proof we will denote the r-component of $\widehat{\sigma}_i$ as $\mathfrak{C}_r^i$ for i, $1 \le i \le k-1$. In case $i = 1$ we also denote $\mathfrak{C}_r = \mathfrak{C}_r^1$ and in case $i = k-1$ we denote $\mathfrak{C}_r^* = \mathfrak{C}_r^{k-1}$.

Note that we have the following equality

$$\sum_{r>1} r \cdot |\mathfrak{C}_r| = supp(\omega) = 3k \tag{5.3}$$

and as a consequence for any $r > 1$

$$r \cdot |\mathfrak{C}_r| \le 3k. \tag{5.4}$$

According to Lemma 5.2 there must be some r-component of $\widehat{\sigma}_1$ of length $\geq k-2$. But if $|\mathfrak{C}_r| \geq k-2$ then by (5.4) we have

$$r \le \frac{3k}{|\mathfrak{C}_r|} \le \frac{3k}{k-2} = 3 + \frac{6}{k-2} < 4 \quad \text{for } k > 8 \tag{5.5}$$

Hence r is 2 or 3 and we now proceed according to these two possibilities.

Case 1 : $|\mathfrak{C}_2| \geq \mathbf{k-2}$

We will show that this case is impossible.

First we claim that $|\mathfrak{C}_r| < k-2$ for each $r > 2$. For $r > 3$ this follows from (5.5) and for $r = 3$, if $|\mathfrak{C}_3| \geq k-2$, we would have that

$$3 \cdot |\mathfrak{C}_3| + 2 \cdot |\mathfrak{C}_2| \geq 3(k-2) + 2(k-2) = 5k - 10 > 3k \qquad \text{for} \quad k > 8$$

contradicting (5.3).

Hence we conclude from Theorem 3.3 (e) that for each r-component $\mathfrak{C}_r$ where $r > 2$, the retraction of ω to $\mathfrak{C}_r$, $\Omega_{\mathfrak{C}_r} \colon B_{k-2} \to \mathbf{S}(\mathfrak{C}_r) \cong S_{|\mathfrak{C}_r|}$, is cyclic (since $|\mathfrak{C}_r| < k-2$). Put $\Sigma_{\mathfrak{C}_r} = supp(\mathfrak{C}_r)$. According to Corollary 2.17 the reduction of ω corresponding to each such component $\mathfrak{C}_r$, $\omega_{\mathfrak{C}_r} \colon B_{k-2} \to \mathbf{S}(supp(\mathfrak{C}_r))$, is also cyclic. This means that $\omega_{\Sigma_{\mathfrak{C}_r}}(\sigma_3) = \ldots = \omega_{\Sigma_{\mathfrak{C}_r}}(\sigma_{k-1})$, and thus

$$\widehat{\sigma}_3|\Sigma_{\mathfrak{C}_r} = \ldots = \widehat{\sigma}_{k-1}|\Sigma_{\mathfrak{C}_r}. \tag{5.6}$$

Put $\Sigma = \bigcup_{\mathfrak{C}_r} \Sigma_{\mathfrak{C}_r}$, where $\mathfrak{C}_r$ runs over all the r-components of $\widehat{\sigma}_1$ for $r > 2$. The set Σ is invariant under all the permutations $\widehat{\sigma}_3, ..., \widehat{\sigma}_{k-1}$. Since (5.6) holds for every r-component $\mathfrak{C}_r$ of $\widehat{\sigma}_1$ for $r > 2$, it follows that there is a permutation $S \in \mathbf{S}(\Sigma)$ such that $\widehat{\sigma}_3|\Sigma = \ldots = \widehat{\sigma}_{k-1}|\Sigma = S$.

Consider now $\mathfrak{C}_2$. Denote $\Sigma' = supp(\mathfrak{C}_2)$. There are two possibilities for $\Omega_{\mathfrak{C}_2}$ (the retraction of ω to $\mathfrak{C}_2$) : It is either cyclic or non-cyclic. If $\Omega_{\mathfrak{C}_2}$ were cyclic, we would get that the reduction of ω corresponding to Σ' would be cyclic by Corollary 2.17 and hence there would be a permutation $S' \in \mathbf{S}(\Sigma')$ such that $\widehat{\sigma}_3|\Sigma' = \ldots = \widehat{\sigma}_{k-1}|\Sigma' = S'$, but since $\Sigma \cup \Sigma' = \Delta_{3k} = supp(\widehat{\sigma}_i)$ for each $i = 3, \ldots, k-1$, this would mean that $\widehat{\sigma}_3 = \ldots = \widehat{\sigma}_{k-1} = SS'$ which would imply by Lemma 2.10 that ω is cyclic which contradicts our assumption.

Hence assume that $\Omega_{\mathfrak{C}_2}$ is non-cyclic. Note that $|\mathfrak{C}_2| < 2(k-2)$ for otherwise

$$2 \cdot |\mathfrak{C}_2| \geq 2 \cdot 2(k-2) = 4k - 8 > 3k \quad \text{for } k > 8$$

contradicting (5.4). Hence $k - 2 \leq |\mathfrak{C}_2| < 2(k-2)$ and we may conclude from Theorem 3.3 (c) and (d) that $\Omega_{\mathfrak{C}_2} \colon B_{k-2} \to \mathbf{S}(\mathfrak{C}_2) \cong S_{|\mathfrak{C}_2|}$ is conjugate to $\mu_{k-2} \times \kappa$ where κ is cyclic (μ_{k-2} is the canonical epimorphism, see Definition 2.7). This means that there is a 2-subcomponent, $\mathfrak{C}'_2$, of $\widehat{\sigma}_1$ such that κ is the retraction of ω to $\mathfrak{C}'_2$. Since κ is cyclic then according to Corollary 2.17, the reduction of ω corresponding to the component $\mathfrak{C}'_2$, $\omega_{\mathfrak{C}'_2} \colon B_{k-2} \to \mathbf{S}(supp(\mathfrak{C}'_2))$, is also cyclic. This means that $\omega_{\Sigma_{\mathfrak{C}'_2}}(\sigma_3) = \ldots = \omega_{\Sigma_{\mathfrak{C}'_2}}(\sigma_{k-1})$, and thus

$$\widehat{\sigma}_3|\Sigma_{\mathfrak{C}'_2} = \ldots = \widehat{\sigma}_{k-1}|\Sigma_{\mathfrak{C}'_2}. \tag{5.7}$$

Now, the set $\Sigma \cup supp(\mathfrak{C}'_2)$ is invariant under all the permutations $\widehat{\sigma}_3, ..., \widehat{\sigma}_{k-1}$, and hence , by combining (5.7) and (5.6) it follows that there is a permutation $T' \in \mathbf{S}(\Sigma \cup supp(\mathfrak{C}'_2))$ such that $\widehat{\sigma}_3|\Sigma \cup supp(\mathfrak{C}'_2) = \ldots = \widehat{\sigma}_{k-1}|\Sigma \cup supp(\mathfrak{C}'_2) = T'$.

Denote $\mathfrak{C}_2'' = \mathfrak{C}_2 - \mathfrak{C}_2'$ and set $T_i'' = \widehat{\sigma}_i | supp(\mathfrak{C}_2''),\ \ i = 3, ..., k-1$. We now have

$$\widehat{\sigma}_i = T' \cdot T_i'' \qquad \text{for all} \ \ i = 3, ..., k-1. \tag{5.8}$$

where

$$supp(T') \cap supp(T_i'') = \emptyset, \quad |supp(T')| = k+4 \quad \text{and} \quad |supp(T_i'')| = 2(k-2) \tag{5.9}$$

for each $i = 3, ..., k-1$.

Consider the reduction of ω to $supp(\mathfrak{C}_2'')$, $\omega_{supp(\mathfrak{C}_2'')} \colon B_{k-2} \to \mathbf{S}(supp(\mathfrak{C}_2'')) \cong S_{2(k-2)}$. Denote $\phi = \omega_{supp(\mathfrak{C}_2'')}$. According to the above arguments ϕ is not cyclic and the degenerate component of $\phi(\sigma_i)$ is empty for each $i = 3, \ldots, k-1$. Hence, according to Lemma 7.28 in [Lin04b], ϕ is standard and is conjugate to either of the model homomorphisms φ_2 or φ_3 as defined in Definition 3.1.

Since $\widehat{\sigma}_1$ and $\widehat{\sigma}_{k-1}$ are conjugate, they have the same cycle structure. Hence we can argue in a completely similar way on the components of $\widehat{\sigma}_{k-1}$ (instead of $\widehat{\sigma}_1$) to conclude that there are permutations $R', R_i'' \in S_{3k}$ such that

$$\widehat{\sigma}_i = R' \cdot R_i'' \qquad \text{for all} \ \ i = 1, ..., k-3. \tag{5.10}$$

where

$$supp(R') \cap supp(R_i'') = \emptyset, \quad |supp(R')| = k+4 \quad \text{and} \quad |supp(R_i'')| = 2(k-2) \tag{5.11}$$

for each $i = 3, ..., k-1$.

By comparing (5.8) and (5.10) we have that

$$\widehat{\sigma}_i = T' \cdot T_i'' = R' \cdot R_i'' \qquad \text{for all} \ \ i = 3, ..., k-3. \tag{5.12}$$

In particular

$$T_i''|_{supp(T_i'')} = R'|_{supp(T_i'')} \cdot R_i''|_{supp(T_i'')} \qquad i = 3, \ldots, k-3. \tag{5.13}$$

Since $\phi \colon B_{k-2} \to \mathbf{S}(supp(\mathfrak{C}_2'')) \cong S_{2(k-2)}$, where $\phi(\sigma_i) = T_i''|_{supp(T_i'')}$, is conjugate to either φ_2 or φ_3 and since $supp(R') \cap supp(R_i'') = \emptyset$ by (5.11) , we get from (5.13), according to Lemma 3.2, that $|supp(R'|_{supp(T_i'')})| \leq 4$, i.e., $|supp(R') \cap supp(T_i'')| \leq 4$. Hence, by using (5.9) and (5.11), we have

$$k+4=|supp(R')|=|supp(R')\cap\Delta_{3k}|=|supp(R')\cap(supp(T')\cup supp(T_i''))|=$$

$$=|(supp(R')\cap supp(T'))\cup(supp(R')\cap supp(T_i''))|\le$$

$$\le|(supp(R')\cap supp(T'))|+|(supp(R')\cap supp(T_i''))|\le|(supp(R')\cap supp(T'))|+4$$

And so $|(supp(R')\cap supp(T'))|\ge k$.

Now according to (5.12), the set $\Psi'=supp(R')\cap supp(T')$ is a union of supports of cycles in each $\widehat{\sigma}_i,\ i=3,\ldots,k-3$ and so there exists a non-trivial permutation $U\in\mathbf{S}(supp(R')\cap supp(T'))$ such that

$$R'|_{\Psi'}=T'|_{\Psi'}=U$$

which means that

$$\widehat{\sigma}_i=U\cdot U_i\qquad\text{for all}\ \ i=1,...,k-1.\tag{5.14}$$

where $supp(U)\cap supp(U_i)=\emptyset$ for $i=1,\ldots,k-1$ and so ω is not transitive which contradicts our assumption.

Case 2 : $|\mathfrak{C}_3|\ge\mathbf{k-2}$

We will divide this case to the three possibilities : $|\mathfrak{C}_3|=\mathbf{k-1},\mathbf{k},$ or $\mathbf{k-2}$ respectively.

Subcase 2.1 : $|\mathfrak{C}_3|=\mathbf{k-1}$

We will show that this case is impossible.

Suppose $|\mathfrak{C}_3|=k-1$. Let us write (5.3) in this case :

$$3\cdot|\mathfrak{C}_3|+\sum_{r\ne3,r>1}r\cdot|\mathfrak{C}_r|=3(k-1)+\sum_{r\ne3,r>1}r\cdot|\mathfrak{C}_r|=3k$$

and this implies the equation

$$\sum_{r\ne3,r>1}r\cdot|\mathfrak{C}_r|=3$$

which has no solutions.

Subcase 2.2 : $|\mathfrak{C}_3| = \mathbf{k}$

We will show that this case is impossible as well.

Denote $\Sigma = supp(\mathfrak{C}_3)$. There are two possibilities for $\Omega_{\mathfrak{C}_3}$ (the retraction of ω to $\mathfrak{C}_3$) : It is either cyclic or non-cyclic. If $\Omega_{\mathfrak{C}_3}$ were cyclic, we would get that the reduction of ω corresponding to Σ would be cyclic by Corollary 2.17 and hence there would be a permutation $S \in \mathbf{S}(\Sigma)$ such that $\widehat{\sigma}_3|\Sigma = \ldots = \widehat{\sigma}_{k-1}|\Sigma = S$, but since $\Sigma = \Delta_{3k}$ in this case, this would mean that $\widehat{\sigma}_3 = \ldots = \widehat{\sigma}_{k-1} = S$ which would imply by Lemma 2.10 that ω is cyclic which contradicts our assumption.

Hence we can assume that $\Omega_{\mathfrak{C}_3}$ is non-cyclic. Since $|\mathfrak{C}_3| = k$ we have by Theorem 3.3 (c) that $\Omega_{\mathfrak{C}_3} \colon B_{k-2} \to \mathbf{S}(\mathfrak{C}_3) \cong S_k$ (the retraction of ω to the 3-component $\mathfrak{R}$) is conjugate to $\mu_{k-2} \times \kappa$ where κ is cyclic and $\mu_{k-2} \colon B_{k-2} \to S_{k-2}$ is the canonical epimorphism (see Definition 2.7). In particular, this means that there are $C_1, C_2 \in \mathfrak{C}_3$ such that

$$C_1^{\widehat{\sigma}_3} = C_2 \qquad \text{and} \qquad C_2^{\widehat{\sigma}_3} = C_1.$$

But according to Lemma 3.5 and Example 3.8, $\widehat{\sigma}_3|_{supp(C_1) \cup supp(C_2)}$ must be of cyclic type $[2,2,2]$ or $[6]$ which is impossible in this case.

Subcase 2.3 : $|\mathfrak{C}_3| = \mathbf{k - 2}$

We have shown that this is the only possible value of $|\mathfrak{C}_3|$. We will now prove that in this case ω is indeed conjugate to a model homomorphism, as defined in Definition 3.9.

According to (5.3) we have in this case

$$3 \cdot |\mathfrak{C}_3| + \sum_{r>1, r\neq 3} r \cdot |\mathfrak{C}_r| = 3k$$

or (since $|\mathfrak{C}_3| = k - 2$)

$$\sum_{r>1, r\neq 3} r \cdot |\mathfrak{C}_r| = 6$$

and in particular for any $r > 1, r \neq 3$:

$$r \cdot |\mathfrak{C}_r| \leq 6$$

which implies that for any $r > 1, r \neq 3$:

$$|\mathfrak{C}_r| \leq 3. \tag{5.15}$$

Hence, according to Theorem 3.3 (e), the retraction of ω to $\mathfrak{C}_r$ for any $r > 1, r \neq 3$, $\Omega_{\mathfrak{C}_r} \colon B_{k-2} \to \mathbf{S}(\mathfrak{C}_r) \cong S_{|\mathfrak{C}_r|}$, is cyclic since $k-2 > 3$ (for $k > 8$) and $|\mathfrak{C}_r| \leq 3$. It follows from Corollary 2.17 that the reduction of ω corresponding to $\mathfrak{C}_r$ for any $r > 1, r \neq 3$ is cyclic. We claim that $\Omega_{\mathfrak{C}_3}$, the retraction of ω to $\mathfrak{C}_3$, is non-cyclic. For suppose $\Omega_{\mathfrak{C}_3}$ were cyclic, then according to Corollary 2.17 the reduction of ω corresponding to $\mathfrak{C}_3$, $\omega_{supp(\mathfrak{C}_3)}$, would be cyclic. Now since

$$supp(\mathfrak{C}_3) \cup \bigcup_{r>1, r\neq 3} supp(\mathfrak{C}_r) = \bigcup_{r>1} supp(\mathfrak{C}_r) = supp(\omega)$$

this would mean that $\omega|_{supp(\omega)} = \omega$ is cyclic on B_{k-2}, i.e., $\omega(\sigma_3) = \cdots = \omega(\sigma_{k-1})$ and by lemma 2.10 this means that ω itself is cyclic which contradicts our assumption.

Hence we can assume that $\Omega_{\mathfrak{C}_3} \colon B_{k-2} \to \mathbf{S}(\mathfrak{C}_3) \cong S_{k-2}$ is non-cyclic. According to Theorem 3.3 (d), this means that $\Omega_{\mathfrak{C}_3}$ is conjugate to μ_{k-2}, the canonical epimorphism (see Definition 2.7). Since $\widehat{\sigma}_1$ and $\widehat{\sigma}_{k-1}$ are conjugate, they have the same cycle structure and we can argue as above on the r-components of $\widehat{\sigma}_{k-1}$ to deduce that $\Omega_{\mathfrak{C}_3^*}$ is conjugate to μ_{k-2} as well.

By conjugating ω with an appropriate permutation we may assume w.l.o.g that $\mathfrak{C}_3^* = \{A_1, \ldots, A_{k-2}\}$ (see Notation 5.1). Furthermore, by Lemma 5.4 we can assume that $\Omega_{\mathfrak{C}_3^*} = \mu_{k-2}$ (and not just $\Omega_{\mathfrak{C}_3^*} \sim \mu_{k-2}$) so that

$$A_i^{\widehat{\sigma}_j} = \begin{cases} A_{i+1} & i = j \\ A_{i-1} & i = j+1 \\ A_i & i \neq j, j+1 \end{cases} \qquad 1 \leq j \leq k-3,\ 1 \leq i \leq k-2 \tag{5.16}$$

In particular we see that for each i, $1 \leq i \leq k-3$, $\widehat{\sigma}_i$ *induces a transposition* between A_i and A_{i+1}, i.e.,

$$A_i^{\widehat{\sigma}_i} = A_{i+1} \quad \text{and} \quad A_{i+1}^{\widehat{\sigma}_i} = A_i. \tag{5.17}$$

Hence,

$$supp(A_i) \cup supp(A_{i+1}) \in Inv(\widehat{\sigma}_i). \tag{5.18}$$

Denote $C_i = \widehat{\sigma}_i|_{supp(A_i) \cup supp(A_{i+1})}$. Then (5.17) and (5.18) mean that each cycle in the cyclic decomposition of C_i is included in the cyclic decomposition of $\widehat{\sigma}_i$ and we can write

$$A_i^{C_i} = A_{i+1} \quad \text{and} \quad A_{i+1}^{C_i} = A_i. \tag{5.19}$$

According to Lemma 3.5 and as explained in Example 3.8, the cyclic type of C_i must be either $[2,2,2]$ or $[6]$ and so, it is disjoint from $\mathfrak{C}_3^i$. This means that the cyclic type of each $\widehat{\sigma}_i$ has only two possibilities :

$$[\underbrace{3,\ldots,3}_{k-2\ times},2,2,2] \qquad \text{or} \qquad [\underbrace{3,\ldots,3}_{k-2\ times},6] \tag{5.20}$$

Furthermore, from (5.16) we get that

$$A_i^{\widehat{\sigma}_j} = A_i \qquad \text{for } 1 \le i \le k-2,\ 1 \le j \le k-3,\ i \ne j, j+1. \tag{5.21}$$

But since $supp(A_i) \subset supp(\widehat{\sigma}_j)$ for these indices we have, by Lemma 5.3 (b), that $\widehat{\sigma}_j|_{supp(A_i)} = A_i^{\pm 1}$.

So far we have concluded that (up to conjugation) $\omega|_{supp(\mathfrak{C}_3^*)}\colon B_{k-2}^* \to \mathbf{S}(supp(\mathfrak{C}_3^*)) \cong S_{3(k-2)}$ satisfies

$$\widehat{\sigma}_i|_{supp(\mathfrak{C}_3^*)} = A_1^{\pm 1} \cdots A_{i-1}^{\pm 1} \cdot C_i \cdot A_{i+2}^{\pm 1} \cdots A_{k-2}^{\pm 1} \qquad \text{for } i = 1, \ldots, k-3 \tag{5.22}$$

Let us use the following notation :

$$\mathfrak{C}_3^i = \{D_1^i, \ldots, D_{i-1}^i, D_{i+2}^i, \ldots, D_k^i\} \qquad 1 \le i \le k-1 \tag{5.23}$$

where $D_j^i = A_j^{\pm 1}$ for $1 \le j \le k-2,\ 1 \le i \le k-3$ (this can be assumed according to (5.22)) and $D_j^{k-1} = A_j$ for $1 \le j \le k-2$ (this is so by assumption).

We will prove the rest of our claim in a series of steps, as follows.

Step 1: Up to conjugation in $S_{\Delta_{3k} - supp(\mathfrak{C}_3^*)}$ (a conjugation that does not alter (5.22)), ω satisfies the following

$$\widehat{\sigma}_i|_{\Delta_{3k} - supp(\mathfrak{C}_3^*)} = A_{k-1} \cdot A_k \qquad \text{for} \quad 1 \le i \le k-3$$

and if we write

$$\widehat{\sigma}_{k-1} = A_1 \cdots A_{k-2} \cdot C_{k-1}$$

then

$$A_{k-1}^{C_{k-1}} = A_k \qquad \text{and} \qquad A_k^{C_{k-1}} = A_{k-1}$$

Proof of Step 1:

Since $\widehat{\sigma}_1$ and $\widehat{\sigma}_{k-1}$ are conjugate, they have the same cycle structure. Hence, from (5.15) we have that $|\mathfrak{C}_r^*| \leq 3$ for any $r > 1, r \neq 3$. In fact, by (5.20) we know that $\mathfrak{C}_r^* \neq \emptyset$ for $r > 1, r \neq 3$ only if r is equal either to 2 or 6. Let s be that value such that $s > 1, s \neq 3$ and $\mathfrak{C}_s^* \neq \emptyset$.

Hence, according to Theorem 3.3 (e), the coretraction of ω to $\mathfrak{C}_s^*$, $\Omega_{\mathfrak{C}_s^*} \colon B_{k-2}^* \to \mathbf{S}(\mathfrak{C}_s^*) \cong S_{|\mathfrak{C}_s^*|}$, is cyclic since $k-2 > 3$ (for $k > 8$) and $|\mathfrak{C}_s^*| \leq 3$. It follows by Corollary 2.17 that the coreduction of ω corresponding to $\mathfrak{C}_s^*$, $\omega_{supp(\mathfrak{C}_s^*)}$, is cyclic.

Now since

$$supp(\mathfrak{C}_s^*) = \Delta_{3k} - \bigcup_{i=1,\ldots,k-2} supp(A_i) = \Delta_{3k} - supp(\mathfrak{C}_3^*)$$

and since, by (5.22), there are $k-4$ 3-cycles in the cyclic decomposition of $\widehat{\sigma}_i|_{supp(\mathfrak{C}_3^*)}$ for $1 \leq i \leq k-3$, this means that there are two (disjoint) 3-cycles, $D_{k-1}, D_k \in S_{supp(\mathfrak{C}_s^*)}$, such that

$$\widehat{\sigma}_i|_{\Delta_{3k}-supp(\mathfrak{C}_3^*)} = D_{k-1} \cdot D_k \qquad \text{for} \quad 1 \leq i \leq k-3 \tag{5.24}$$

Putting $i = 1$ in (5.24) and using notation (5.23) we can write in particular $D_{k-1} = D_{k-1}^1$ and $D_k = D_k^1$.

Now again by (5.22) we have that

$$(D_i^1)^{\widehat{\sigma}_{k-1}|_{supp(\mathfrak{C}_3^*)}} = D_i^1 \qquad \text{for} \quad i = 3, \ldots, k-2$$

and if we write

$$\widehat{\sigma}_{k-1} = A_1 \cdots A_{k-2} \cdot C_{k-1}$$

i.e., $\widehat{\sigma}_{k-1}|_{supp(C_s^*)} = C_{k-1}$ then since $\Omega_{\mathfrak{C}_3} \sim \mu_{k-2}$ we necessarily have that

$$(D_{k-1}^1)^{C_{k-1}} = D_k^1 \qquad \text{and} \qquad (D_k^1)^{C_{k-1}} = D_{k-1}^1$$

Finally, by conjugating ω with an appropriate permutation in $S_{\Delta_{3k}-supp(\mathfrak{C}_3^*)} = S_{supp(\mathfrak{C}_s)}$ we can assume that $D_{k-1} = A_{k-1}$ and $D_k = A_k$. In terms of notation (5.23) this means that $D_{k-1}^i = A_{k-1}$ and $D_k^i = A_k$ for each $i = 1, \ldots, k-3$.

□

Step 2: There are only two possibilities : Either

$$D_j^1 = A_j \qquad \text{for } j = 3, \ldots, k-2$$

or

$$D_j^1 = A_j^{-1} \qquad \text{for } j = 3, \ldots, k-2$$

(see the notation in (5.23))

Proof of Step 2:

Using the notation in (5.23) and using step 1 and (5.22) we have that

$$\widehat{\sigma}_i = D_1^i \cdots D_{i-1}^i \cdot C_i \cdot D_{i+2}^i \cdots D_{k-2}^i A_{k-1} A_k \qquad \text{for} \quad i = 1, \ldots, k-3$$

where $D_j^i = A_j^{\pm 1}$. Hence we have by (5.19)

$$(D_i^1)^{\widehat{\sigma}_i} = (D_i^1)^{C_i} = D_{i+1}^1 \qquad \text{for} \quad i = 3, \ldots, k-3$$

But according to (5.19)

$$A_i^{C_i} = A_{i+1} \qquad \text{for} \quad i = 3, \ldots, k-3$$

which implies that

$$(A_i^{-1})^{C_i} = A_{i+1}^{-1} \qquad \text{for} \quad i = 3, \ldots, k-3$$

Hence, for each $i = 3, \ldots, k-3$, we have that $D_i^1 = A_i$ iff $D_{i+1}^1 = A_{i+1}$. And so, either $D_i^1 = A_i$ for all $i = 3, \ldots, k-3$ or $D_i^1 = A_i^{-1}$ for all $i = 3, \ldots, k-3$.

□

Step 3: For each i, $1 \leq i \leq k-3$, we have that (using the notation in (5.23))

$$D_j^i \in \mathfrak{C}_3^i,\ D_j^{i+1} \in \mathfrak{C}_3^{i+1} \quad \text{implies that}$$

$$D_j^i = A_j \qquad \text{iff} \qquad D_j^{i+1} = A_j$$

For any j, $1 \leq j \leq k-3$.

Proof of Step 3:

Suppose $D_j^i \in \mathfrak{C}_3^i,\ D_j^{i+1} \in \mathfrak{C}_3^{i+1}$ for some i, $1 \leq i \leq k-3$. Recall (see (5.23)) that $D_j^i, D_j^{i+1} = A_j^{\pm 1}$.

Since D_j^i is a cycle in the cyclic decomposition of $\widehat{\sigma}_i$ for each $j \neq i, i+1$, we have that

$$(D_j^i)^{\widehat{\sigma}_{i+1}\widehat{\sigma}_i} \in \mathfrak{C}_3^{i+1} \tag{5.25}$$

But since

$$supp(D_j^i) = supp(D_j^{i+1}) \in Inv(\widehat{\sigma}_i) \cap Inv(\widehat{\sigma}_{i+1}) \tag{5.26}$$

we have that

$$(D_j^i)^{\widehat{\sigma}_{i+1}\widehat{\sigma}_i} = (D_j^i)^{D_j^{i+1} D_j^i} = D_j^{i+1} \tag{5.27}$$

Now suppose $D_j^i = A_j$, then

$$(D_j^i)^{D_j^{i+1} D_j^i} = A_j^{A_j^{\pm 1} A_j} = A_j$$

which means by (5.27) that $D_j^{i+1} = A_j$. Similarly, $D_j^i = A_j^{-1}$ implies that $D_j^{i+1} = A_j^{-1}$.

$\square$

Step 4: For each i, $1 \leq i \leq k-4$, we have

$$D_{i+2}^i = A_{i+2} \qquad \text{iff} \qquad D_i^{i+1} = A_i$$

Proof of Step 4:

Rewriting (5.22) in terms of the notation in (5.23) we have

$$\widehat{\sigma}_i|_{supp(\mathfrak{C}_3^*)} = D_1^i \cdots D_{i-1}^i \cdot C_i \cdot D_{i+2}^i \cdots D_{k-2}^i \qquad \text{for } i = 1, \ldots, k-3 \tag{5.28}$$

where $D_j^i = A_j^{\pm 1}$ for any j, $1 \leq j \leq k-2$.

Since D_j^i is a cycle in the cyclic decomposition of $\widehat{\sigma}_i$ for each i, we have that

$$(D^i_{i+2})^{\widehat{\sigma}_{i+1}\widehat{\sigma}_i} \in \mathfrak{C}^{i+1}_3 \qquad 1 \le i \le k-3 \tag{5.29}$$

But since

$$supp(D^i_{i+2}) \cup supp(C_i) = supp(D^{i+1}_i) \cup supp(C_{i+1}) \in Inv(\{\widehat{\sigma}_i, \widehat{\sigma}_{i+1}\})$$

and since the cyclic type of each C_i is either $[2,2,2]$ or $[6]$ (and in any case not $[3,3]$, see (5.20)), (5.29) implies

$$(D^i_{i+2})^{\widehat{\sigma}_{i+1}\widehat{\sigma}_i} = (D^i_{i+2})^{D^{i+1}_i C_{i+1} D^i_{i+2} C_i} = D^{i+1}_i \tag{5.30}$$

Now assume that $D^i_{i+2} = A_{i+2}$, then

$$(D^i_{i+2})^{D^{i+1}_i C_{i+1} D^i_{i+2} C_i} = (A_{i+2})^{A^{\pm 1}_i C_{i+1} A_{i+2} C_i} = A_i$$

where we used (5.19) in the last equality. So (5.30) implies that $D^{i+1}_i = A_i$ in this case. Similarly, $D^i_{i+2} = A^{-1}_{i+2}$ implies that $D^{i+1}_i = A^{-1}_i$.

□

Step 5: There are only two possibilities : Either

$$D^i_j = A_j \qquad \text{for all} \qquad 1 \le i \le k-3,\ 1 \le j \le k-2,\ j \ne i, i+1$$

or

$$D^i_j = A^{-1}_j \qquad \text{for all} \qquad 1 \le i \le k-3,\ 1 \le j \le k-2,\ j \ne i, i+1$$

Proof of Step 5:

Rewriting (5.22) in terms of the notation in (5.23) we have

$$\widehat{\sigma}_i|_{supp(\mathfrak{C}^*_3)} = D^i_1 \cdots D^i_{i-1} \cdot C_i \cdot D^i_{i+2} \cdots D^i_{k-2} \qquad \text{for } i = 1, \ldots, k-3 \tag{5.31}$$

Let $D^i_j \in \mathfrak{C}^i_3$ for some i, $1 \le i \le k-3$. By (5.31) there are two possibilities : Either $j \ge i+2$ or $j \le i-1$. Assume first that $j \ge i+2$.

According to step 3 and by (5.31) we have that

$$D_j^i = A_j \quad \text{iff} \quad D_j^{i-1} = A_j \quad \text{iff} \quad \cdots \quad \text{iff} \quad D_j^1 = A_j \tag{5.32}$$

Now assume that $j \leq i-1$.

According to step 3 and (5.31) again we have that

$$D_j^i = A_j \quad \text{iff} \quad D_j^{i-1} = A_j \quad \text{iff} \quad \cdots \quad \text{iff} \quad D_j^{j+1} = A_j \tag{5.33}$$

By step 4 we have that

$$D_j^{j+1} = A_j \quad \text{iff} \quad D_{j+2}^j = A_{j+2} \tag{5.34}$$

and again by step 3

$$D_{j+2}^j = A_{j+2} \quad \text{iff} \quad D_{j+2}^{j-1} = A_{j+2} \quad \text{iff} \quad \cdots \quad \text{iff} \quad D_{j+2}^1 = A_{j+2} \tag{5.35}$$

Summing up, we have by (5.32) that for each i, $1 \leq i \leq k-3$

$$D_j^i = A_j \quad \text{iff} \quad D_j^1 = A_j \qquad \text{for any} \quad j \geq i+2 \tag{5.36}$$

and by (5.33), (5.34) and (5.35)

$$D_j^i = A_j \quad \text{iff} \quad D_{j+2}^1 = A_{j+2} \qquad \text{for any} \quad j \leq i-1 \tag{5.37}$$

But according to step 2

$$D_j^1 = A_j \quad \text{iff} \quad D_3^1 = A_3 \qquad \text{for any} \quad 3 \leq j \leq k-2 \tag{5.38}$$

Hence (5.36), (5.37) and (5.38) imply that

$$D_j^i = A_j \quad \text{iff} \quad D_3^1 = A_3 \qquad \text{for any} \quad 1 \leq i \leq k-3, \quad 1 \leq j \leq k-2, j \neq i, i+1$$

Since $D_j^i = A_j^{\pm 1}$ this implies the claim.

$\square$

Step 6:

$$D_j^i = A_j \qquad \text{for all} \qquad 1 \leq i \leq k-3,\ 1 \leq j \leq k-2,\ j \neq i, i+1$$

and

$$\widehat{\sigma}_i|_{\Delta_{3k} - supp(\mathfrak{C}_3)} = A_1 \cdot A_2 \qquad \text{for} \quad 3 \leq i \leq k-1$$

Proof of Step 6:

From (5.15) we have that $|\mathfrak{C}_r| \leq 3$ for any $r > 1, r \neq 3$. In fact, by (5.20) we know that $\mathfrak{C}_r \neq \emptyset$ for $r > 1, r \neq 3$ only if r is equal either to 2 or 6. Let s be that value such that $s > 1, s \neq 3$ and $\mathfrak{C}_s \neq \emptyset$.

Hence, according to Theorem 3.3 (e), the retraction of ω to $\mathfrak{C}_s$, $\Omega_{\mathfrak{C}_s} \colon B_{k-2} \to \mathbf{S}(\mathfrak{C}_s) \cong S_{|\mathfrak{C}_s|}$, is cyclic since $k-2 > 3$ (for $k > 8$) and $|\mathfrak{C}_s| \leq 3$. It follows from Corollary 2.17 that the reduction of ω corresponding to $\mathfrak{C}_s$, $\omega_{supp(\mathfrak{C}_s)}$, is cyclic.

Now since

$$supp(\mathfrak{C}_s) = \Delta_{3k} - \bigcup_{i=3,\ldots,k} supp(A_i) = \Delta_{3k} - supp(\mathfrak{C}_3)$$

and since, by (5.22), there are $k-4$ 3-cycles in the cyclic decomposition of $\widehat{\sigma}_i|_{supp(\mathfrak{C}_3)}$ for $1 \leq i \leq k-3$ and hence also for $i = k-2$ and $i = k-1$, this means that there are two (disjoint) 3-cycles, $D_1, D_2 \in S_{supp(\mathfrak{C}_s)}$, such that

$$\widehat{\sigma}_i|_{\Delta_{3k} - supp(\mathfrak{C}_3)} = D_1 \cdot D_2 \qquad \text{for} \quad 3 \leq i \leq k-1$$

But we already have that

$$\widehat{\sigma}_{k-1}|_{\Delta_{3k} - supp(\mathfrak{C}_3)} = A_1 \cdot A_2$$

Hence

$$\widehat{\sigma}_i|_{\Delta_{3k} - supp(\mathfrak{C}_3)} = A_1 \cdot A_2 \qquad \text{for} \quad 3 \leq i \leq k-1$$

In particular, this means that $D_1^{k-3} = A_1$ and by step 5 this implies that

$$D_j^i = A_j \qquad \text{for all} \qquad 1 \leq i \leq k-3,\ 1 \leq j \leq k-2,\ j \neq i, i+1$$

$\square$

By combining (5.22), step 1 and step 6 we have established that up to conjugation, ω satisfies

$$\widehat{\sigma}_i = A_1 \cdots A_{i-1} \cdot C_i \cdot A_{i+2} \cdots A_k \qquad \text{for } i = 1, \ldots, k-3 \text{ and } i = k-1 \tag{5.39}$$

where

$$A_i^{C_i} = A_{i+1} \quad \text{and} \quad A_{i+1}^{C_i} = A_i \qquad \text{for } i = 1, \ldots, k-3 \text{ and } i = k-1 \tag{5.40}$$

It is now left to determine $\widehat{\sigma}_{k-2}$ which we do in the next step.

Step 7: Up to conjugation that does not alter (5.39) we can assume that ω satisfies

$$\widehat{\sigma}_{k-2} = A_1 \cdots A_{k-3} C_{k-2} A_k$$

where

$$(A_{k-2})^{C_{k-2}} = A_{k-1} \qquad \text{and} \qquad (A_{k-1})^{C_{k-2}} = A_{k-2}$$

Proof of Step 7:

First, by step 6, we already know that

$$\widehat{\sigma}_{k-2}|_{\Delta_{3k} - supp(\mathfrak{C}_3)} = A_1 \cdot A_2$$

Let us now determine $\widehat{\sigma}_{k-2}|_{supp(\mathfrak{C}_3)}$.

By (5.39) and (5.40), we have that for any i, $1 \leq i \leq k-3$ or $i = k-1$ and any $D_j^1 \in \mathfrak{C}_3$ where $3 \leq j \leq k-1$ and $D_j^1 = A_j$

$$(D_j^1)^{\widehat{\sigma}_i} = (A_j)^{A_1 \cdots A_{i-1} \cdot C_i \cdot A_{i+2} \cdots A_k} = (A_j)^{C_i} = \begin{cases} A_{i+1} & j = i \\ A_i & j = i+1 \\ A_j & j \neq i, i+1 \end{cases} = \begin{cases} D_{i+1}^1 & j = i \\ D_i^1 & j = i+1 \\ D_j^1 & j \neq i, i+1 \end{cases} \tag{5.41}$$

By identifying D_i^1 with $i - 2 \in \Delta_{k-2}$ for $i = 3, \ldots, k$, (5.41) implies that

$$\Omega_{\mathfrak{C}_3}(\sigma_i) = (i, i+1) \in \mathbf{S}(\mathfrak{C}_3) \cong S_{k-2} \qquad \text{for} \quad i = 3, \ldots, k-3 \quad \text{and} \quad i = k-1.$$

Now since $\Omega_{\mathfrak{C}_3} \sim \mu_{k-2}$ this means, by Lemma 5.5, that there are only two possibilities :

$$\Omega_{\mathfrak{C}_3}(\sigma_{k-2}) = (k-2, k-1) \quad \text{or} \quad (k-2, k)$$

Assume first that $\Omega_{\mathfrak{C}_3}(\sigma_{k-2}) = (k-2, k-1)$. This means that for any $D_j^1 \in \mathfrak{C}_3$ where $3 \leq j \leq k$

$$(D_j^1)^{\widehat{\sigma}_{k-2}|_{supp(\mathfrak{C}_3)}} = \begin{cases} D_{k-1}^1 & j = k-2 \\ D_{k-2}^1 & j = k-1 \\ D_j^1 & j \neq k-2, k-1 \end{cases} \tag{5.42}$$

We see that for any $j \neq k-2, k-1$, D_j^1 commutes with $\widehat{\sigma}_{k-2}|_{supp(\mathfrak{C}_3)}$. Since $supp(D_j^1) \subset supp(\mathfrak{C}_3)$ and since D_j^1 is a 3-cycle, we have by Lemma 5.3 (b) that

$$\widehat{\sigma}_{k-2}|_{supp(D_j^1)} = (D_j^1)^{\pm 1} = A_j^{\pm 1} \qquad j \neq k-2, k-1$$

But since $A_j \in \mathfrak{C}_3^*$ for each $j = 3, \ldots, k-3$ and since $A_k \in \mathfrak{C}_3^{k-3}$, we can use step 3 to conclude that

$$\widehat{\sigma}_{k-2}|_{supp(D_j^1)} = (D_j^1) = A_j \qquad j \neq k-2, k-1 \tag{5.43}$$

Furthermore, (5.42) implies that

$$supp(D_{k-2}^1) \cup supp(D_{k-1}^1) = supp(A_{k-2}) \cup supp(A_{k-1}) \in Inv(\widehat{\sigma}_{k-2})$$

so we can denote $C_{k-2} = \widehat{\sigma}_{k-2}|_{supp(D_{k-2}^1) \cup supp(D_{k-1}^1)}$ and by (5.42) we have

$$(D_{k-2}^1)^{C_{k-2}} = D_{k-1}^1 \qquad \text{and} \qquad (D_{k-1}^1)^{C_{k-2}} = D_{k-2}^1$$

or

$$(A_{k-2})^{C_{k-2}} = A_{k-1} \qquad \text{and} \qquad (A_{k-1})^{C_{k-2}} = A_{k-2} \tag{5.44}$$

Combining (5.43) and (5.44) we have that

$$\widehat{\sigma}_{k-2} = A_1 \cdots A_{k-3} C_{k-2} A_k$$

and the claim follows in this case.

Now assume that $\Omega_{\mathfrak{C}_3}(\sigma_{k-2}) = (k-2, k)$. Then in this case we have that for any $D^1_j \in \mathfrak{C}_3$ where $3 \leq j \leq k-1$

$$(D^1_j)^{\widehat{\sigma}_{k-2}|_{supp(\mathfrak{C}_3)}} = \begin{cases} D^1_k & j = k-2 \\ D^1_{k-2} & j = k \\ D^1_j & j \neq k-2, k \end{cases} \tag{5.45}$$

By arguing in a completely similar way to the case $\Omega_{\mathfrak{C}_3}(\sigma_{k-2}) = (k-2, k-1)$ we can conclude that

$$\widehat{\sigma}_{k-2} = A_1 \cdots A_{k-3} A_{k-1} C'_{k-2}$$

where $supp(C'_{k-2}) = supp(A_{k-2}) \cup supp(A_k)$ and

$$(A_{k-2})^{C'_{k-2}} = A_k \qquad \text{and} \qquad (A_k)^{C'_{k-2}} = A_{k-2} \tag{5.46}$$

Recall that

$$(A_{k-1})^{C_{k-1}} = A_k \qquad \text{and} \qquad (A_k)^{C_{k-1}} = A_{k-1} \tag{5.47}$$

Furthermore, since $supp(C_{k-1}) = supp(A_{k-1}) \cup supp(A_k)$, then

$$(A_j)^{C_{k-1}} = A_j \qquad \text{for any} \qquad j \neq k-1, k \tag{5.48}$$

and since $supp(C'_{k-2}) = supp(A_{k-2}) \cup supp(A_k)$, then

$$(A_j)^{C'_{k-2}} = A_j \qquad \text{for any} \qquad j \neq k-2, k \tag{5.49}$$

Let us now conjugate ω by C_{k-1}. By (5.39) and (5.40) we have for any i, $1 \leq i \leq k-3$

$$(\widehat{\sigma}_i)^{C_{k-1}} = (A_1 \cdots A_{i-1} \cdot C_i \cdot A_{i+2} \cdots A_k)^{C_{k-1}} =$$

$$= A_1^{C_{k-1}} \cdots A_{i-1}^{C_{k-1}} \cdot C_i^{C_{k-1}} \cdot A_{i+2}^{C_{k-1}} \cdots A_k^{C_{k-1}} =$$

$$= A_1 \cdots A_{i-1} \cdot C_i \cdot A_{i+2} \cdots A_k = \widehat{\sigma}_i$$

The last equation follows from the fact that $supp(C_i) = supp(A_i) \cup supp(A_{i+1})$ for $1 \leq i \leq k-3$ is disjoint from $supp(C_{k-1}) = supp(A_{k-1}) \cup supp(A_k)$.

For $i = k-1$ we have

$$(\widehat{\sigma}_{k-1})^{C_{k-1}} = (A_1 \cdots A_{k-2} \cdot C_{k-1})^{C_{k-1}} =$$

$$= A_1^{C_{k-1}} \cdots A_{k-2}^{C_{k-1}} \cdot C_{k-1}^{C_{k-1}} = A_1 \cdots A_{k-2} \cdot C_{k-1} = \widehat{\sigma}_{k-1}$$

And finally, for $i = k-2$ we have

$$(\widehat{\sigma}_{k-2})^{C_{k-1}} = (A_1 \cdots A_{k-3} A_{k-1} C'_{k-2})^{C_{k-1}} =$$

$$A_1^{C_{k-1}} \cdots A_{k-3}^{C_{k-1}} A_{k-1}^{C_{k-1}} (C'_{k-2})^{C_{k-1}} =$$

$$= A_1 \cdots A_{k-3} A_k (C'_{k-2})^{C_{k-1}} \tag{5.50}$$

Denote $C_{k-2} = (C'_{k-2})^{C_{k-1}}$. Now (5.50) implies that $supp(C_{k-2}) = supp(A_{k-2}) \cup supp(A_{k-1})$. Furthermore

$$(A_{k-2})^{C_{k-2}} = (A_{k-2})^{C_{k-1}^{-1} C'_{k-2} C_{k-1}} = (A_{k-2})^{C'_{k-2} C_{k-1}} = (A_k)^{C_{k-1}} = A_{k-1}$$

and

$$(A_{k-1})^{C_{k-2}} = (A_{k-1})^{C_{k-1}^{-1} C'_{k-2} C_{k-1}} = (A_k)^{C'_{k-2} C_{k-1}} = (A_{k-2})^{C_{k-1}} = A_{k-2}$$

Summing up, we have shown that conjugating ω by C_{k-1} does not alter (5.39), and $\widehat{\sigma}_{k-2}$ satisfies

$$\widehat{\sigma}_{k-2} = A_1 \cdots A_{k-3} C_{k-2} A_k$$

where

$$(A_{k-2})^{C_{k-2}} = A_{k-1} \qquad \text{and} \qquad (A_{k-1})^{C_{k-2}} = A_{k-2}$$

□

Combining (5.39) and step 7 we have established that, up to conjugation, ω satisfies

$$\widehat{\sigma}_i = A_1 \cdots A_{i-1} \cdot C_i \cdot A_{i+2} \cdots A_k \qquad \text{for } i = 1, \ldots, k-1 \tag{5.51}$$

where

$$A_i^{C_i} = A_{i+1} \quad \text{and} \quad A_{i+1}^{C_i} = A_i \qquad \text{for } i = 1, \ldots, k-1 \tag{5.52}$$

Using Notation 3.6 we can write

$$C_i = C^{(i,i+1),3}_{t_1^i, t_2^i} \qquad \text{for each} \quad i = 1, \ldots, k-1$$

where $t_j^i \in \bigtriangledown_3$.

Since ω is a homomorphism, we have according to Lemma 3.12 (in the case $l = 1$, $m = 3$ in which case $\psi_l(\sigma_i) = \psi_1(\sigma_i) = (i, i+1)$) that

$$C_i \infty C_{i+1} \qquad \text{for} \quad i = 1, \ldots, k-2. \tag{5.53}$$

or equivalently, using Notation 3.6

$$C^{(i,i+1),3}_{t_1^i, t_2^i} \infty C^{(i+1,i+2),3}_{t_1^{i+1}, t_2^{i+1}} \qquad \text{for} \quad i = 1, \ldots, k-2 \tag{5.54}$$

According to Lemma 3.13, (5.54) is equivalent to

$$t_i^i + t_{i+1}^i \equiv t_{i+2}^{i+1} + t_{i+1}^{i+1} (\text{mod } 3) \qquad \text{for} \quad i = 1, \ldots, k-2$$

which is condition (3.10) in Definition 3.9 for $m = 3$ and $l = 1$. Thus we see that (5.51) is a model homomorphism given by (3.11) in Definition 3.9 for the case $m = 3$ and $l = 1$.

We have shown that any good non-cyclic and transitive homomorphism $\omega \colon B_k \to S_{3k}$ is conjugate to a model homomorphism, hence each such homomorphism is standard.

□

6 An Updated Conjecture

It is already known (see Theorem 3.3 (a) through (e)) that any non-cyclic transitive homomorphism $B_k \to S_n$ is good for $n \leq 2k$, $k > 8$. Our first conjecture then is the following

Conjecture 6.1. *Every non-cyclic transitive homomorphism $B_k \to S_n$ for $n > 2k$, $k > 8$, is good (see Definition 3.15).* •

It was conjectured in [MaSu05] that there is no non-cyclic transitive homomorphism $B_k \to S_n$ for $k \not| n$ but as we shall see in Example 6.3 there are an infinite number of refutations for this conjecture. First we define

Definition 6.2. *Let $2 \leq k \in \mathbb{N}$, $1 \leq m \in \mathbb{N}$ and let H be an index n subgroup of S_{mk} where $k \not| n$. Let φ be one of the standard homomorphisms (Definition 3.9) if $m \geq 2$ or the canonical homomorphism (Definition 2.7) if $m = 1$. Then the homomorphism $f \colon B_k \to S_n$ which sends $\sigma_i \in B_k$ to the permutation in S_n that $\varphi(\sigma_i)$ induces on the n cosets of H in S_{mk} by left multiplication is said to be **derived from φ using H**. In general, such a homomorphism f is said to be **derived from a canonical or a standard homomorphism**.* •

Note that a homomorphism $B_k \to S_n$ derived from a canonical or a standard homomorphism is not necessarily transitive nor is it necessarily non-cyclic as the images of the standard homomorphisms do not generate S_{mk} in general. However, we do have

Example 6.3. *In Definition 6.2, we take $m = 1$,*

$$H = \mathbf{S}(\{1, 2\}) \times \mathbf{S}(\{3, \ldots, k\}) \cong S_2 \times S_{k-2}$$

as a $\frac{k(k-1)}{2}$ index subgroup of S_k. Consider the homomorphism $f \colon B_k \to S_{\frac{k(k-1)}{2}}$ which is derived from $\mu_k \colon B_k \to S_k$ using H (see definition 2.7). Then

f is transitive : *First note that μ_k is surjective : $\mu_k(\sigma_i) = (i, i+1)$ which generate S_k for $i = 1, \ldots, k-1$. Hence, for any $g_1, g_2 \in S_k$ there exists an element $g \in B_k$ such that $\mu_k(g) = g_1 g_2^{-1}$. So, for any two cosets of H in S_k, say $g_1 H$ and $g_2 H$, we have that*

$$g_1 H = (g_1 g_2^{-1}) g_2 H = \mu_k(g) g_2 H$$

which means, by definition of f, that f is transitive.

f is non-cyclic : *If f were cyclic then according to Lemma 2.10 and the definition of f, we would have that for each $i = 1, \ldots, k-1$, $\mu_k(\sigma_i) = (i, i+1)$ would induce the same permutation on the cosets of H in S_k by multiplication on the left. However, since $\mu_k(\sigma_1) = (1, 2) \in H$ then $(1, 2)$ sends the coset H to itself while the element $\mu_k(\sigma_2) = (2, 3) \notin H$ sends the coset H to $(2, 3)H \neq H$. Hence f is non-cyclic.*

Finally, note that if k is even then $k \nmid \frac{k(k-1)}{2}$, and so, for any even k, this example indeed refutes the conjecture made in [MaSu05] that there is no non-cyclic transitive homomorphism $B_k \to S_n$ for $k \not| n$.

$\bullet$

We now give the following

Conjecture 6.4. *Every good non-cyclic transitive homomorphism $B_k \to S_n$ is either canonical (if $n = k$), standard (if $k \neq n,\ k \mid n$) or derived from a canonical or a standard homomorphism (if $k \not| n$).* $\bullet$

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